Statistical analysis tolerance using jacobian torsor model based on uncertainty propagation method

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ABSTRACT
One risk inherent in the use of assembly components is that the behaviour of these components is discovered only at the moment an assembly is being carried out. The objective of our work is to enable designers to use known component tolerances as parameters in models that can be used to predict properties at the assembly level. In this paper we present a statistical approach to assemblability evaluation, based on tolerance and clearance propagations. This new statistical analysis method for tolerance is based on the Jacobian-Torsor model and the uncertainty measurement approach. We show how this can be accomplished by modeling the distribution of manufactured dimensions through applying a probability density function. By presenting an example we show how statistical tolerance analysis should be used in the Jacobian-Torsor model. This work is supported by previous efforts aimed at developing a new generation of computational tools for tolerance analysis and synthesis, using the Jacobian-Torsor approach. This approach is illustrated on a simple three-part assembly, demonstrating the method's capability in handling three-dimensional geometry.

Keywords: Simulation; statistical; tolerance analysis; torsor; jacobian.

1. INTRODUCTION

When manufacturing products, it is not possible to obtain the exact dimensions displayed on engineering drawings. The accuracy achieved depends largely on the manufacturing process involved and the care taken to manufacture the product. A tolerance value is thus used to show the manufacturing department the maximum permissible variation from the required dimension. As such, each dimension on a drawing must include a tolerance value, one that appears either as a general value applicable to several dimensions or a specific tolerance value for a particular dimension.

The cost and required precision of assemblies increase according to the technology and performance levels involved. In product design there is thus a great need to pay increased attention to tolerance, in order to enable high precision assemblies to be manufactured at lower costs. Indeed tolerance analysis is a key element for improving product quality in industry. Designers want tight tolerances to assure product performance; while manufacturers prefer loose tolerances in order to reduce costs. There is thus a critical need for a quantitative design tool capable of specifying tolerances. Tolerance analysis can thus combine engineering design requirements and manufacturing capabilities within the same model, in order to quantitatively evaluate the effects of tolerance specifications on both design and manufacturing requirements.

Statistical tolerance analysis is a powerful analytical method because it not only predicts the effect of manufacturing variation on design performance and production cost, but also allows designers and manufacturing personal to take advantage of statistical averaging and thus relax component tolerances without sacrificing quality. In an attempt to apply the Monte Carlo simulation method to tolerance analyses on assemblies, this paper presents a method for analyzing tolerances in dimensional chains. This approach is based on the Jacobian-Torsor model and statistical uncertainty measurement.

2. BIBLIOGRAPHY

The linear “stack-up” method is a fundamental tolerance analysis technique, as described in [1] and others. K.W. Chase at Brigham Young University worked to extend this stack-up analysis to 2- and 3-dimensional assemblies [2,3]. For these analyses he used vector loop-based assembly models, in which closed vector loops describe the small kinematic adjustments and open vector loops describe critical clearances or other assembly features. A commercial software package based on this work is available and can perform worst-case and statistical analyses. Linares [4] creates functional groups of all surfaces that could be found in a joint. These groups are then represented inside the nodes rather than inside the surfaces. In their work, Ballu and Mathieu [5] and Tessandier [6] indicate the joint type on the arcs (e.g. plane joint, revolute joint), Giordano [7] prefers indicating the type of contact, Marguet orientates arcs from the base to the end parts [8], while Wang and Ozsay [9] introduce the notion of subsets into their graphs. For each mechanism configuration, these approaches make use of a graph. The nodes of this graph represent the parts, while the arcs represent the links. A more detailed depiction of the component may be obtained by representing the component surfaces within the graph nodes [10].

Desrochers [11] proposes a 3-D representation of the tolerance zones using small displacement screws to which the parameters of invariance defined by Bourdet and Clément are also applied [12]. The proposed representation makes use of mathematical constraints to define the extreme limits of a tolerance zone in 3-D, and Bourdet [12] develops torsors to mathematically represent tolerances of a system. According to his method, the displacement of a solid in space can be modeled by six constraints that are defined by two 3-vectors, one in translation and the other in rotation. Using these torsors, the outer limits can be obtained by defining a tolerance zone [11].

Monte Carlo simulation is commonly used to obtain a statistical tolerance analysis and synthesis [13]. A random number generator is used to simulate the variability of each component's size and form. These values are then combined through the assembly function to determine the resulting influence on certain clearance or gap dimensions. This simulation method can be quite time-consuming.

The model allows both analysis and synthesis problems to be solved. Laperrière [14] have used open kinematic chains developed in robotics to solve tolerance analysis and synthesis problems. Their model is based on a Jacobian matrix that maps local expressions of each tolerance in a chain into global expressions of the functional requirement. This method has the benefit of being entirely defined from 4×4 transformation matrices readily available in every commercial CAD system. Further developments led to the development of a unified Jacobian–Torsor model [15,16] where interval arithmetic’s [17] allows tolerance intervals to be directly embedded in the various torsors of this model.

3. RELATED WORKS

3.1. JACOBIAN–TORSOR MODEL

The tolerance model variables are derived from different tolerance models, which can be in the form of either conventional plus-minus tolerance representations or geometric tolerance representations. The assembly response system noted by functional requirement (FR) can also be represented in two models, i.e., the closed mathematical model and the relative positioning model. In the closed mathematical model, mathematical equations are formulated; and the variations of design function are directly obtained from the calculation of the equations. In the relative positioning model, an optimization model is used instead of the closed mathematical equations. Using the closed mathematical model, in previous papers a tool for deterministic tolerance analysis was presented [15], [16] which used an interval arithmetic formulation:

$$\begin{bmatrix}[\underline{u},\overline{u}]\\ [\underline{v},\overline{v}]\\ [\underline{w},\overline{w}]\\ [\underline{\alpha},\overline{\alpha}]\\ [\underline{\beta},\overline{\beta}]\\ [\underline{\delta},\overline{\delta}]\end{bmatrix}_{FR} = \Big[\big[J_1 J_2 J_3 J_4 J_5 J_6\big]_{FE1} \ldots\ldots \big[J_1 J_2 J_3 J_4 J_5 J_6\big]_{FEN}\Big] \bullet \begin{bmatrix}\begin{bmatrix}[\underline{u},\overline{u}]\\ [\underline{v},\overline{v}]\\ [\underline{w},\overline{w}]\\ [\underline{\alpha},\overline{\alpha}]\\ [\underline{\beta},\overline{\beta}]\\ [\underline{\delta},\overline{\delta}]\end{bmatrix}_{FE1}\\ \cdots\cdots\cdots\\ \begin{bmatrix}[\underline{u},\overline{u}]\\ [\underline{v},\overline{v}]\\ [\underline{w},\overline{w}]\\ [\underline{\alpha},\overline{\alpha}]\\ [\underline{\beta},\overline{\beta}]\\ [\underline{\delta},\overline{\delta}]\end{bmatrix}_{FEN}\end{bmatrix} \quad (1)$$

where:

$$[FR] = \begin{bmatrix}[\underline{u},\overline{u}]\\ \vdots \\ [\underline{\ddot{a}},\overline{\ddot{a}}]\end{bmatrix}_{FR}$$

and

$$[FEi] = \begin{bmatrix}[\underline{u},\overline{u}]\\ \vdots \\ [\underline{\ddot{a}},\overline{\ddot{a}}]\end{bmatrix}_{FEi}$$

: Small displacements torsors associated to some functional requirement (Play, gap, clearance) represented as a [FR] vector or some Functional Element uncertainties (tolerance, kinematic link, ….) also represented as [FE] vectors; with N representing the number of torsors in a kinematic chain;

$\left[J_1 \cdots J_6\right]_{FEi}$: Jacobian matrix expressing a geometrical relation between a [FR] vector and some corresponding [FE] vector;

$\left(\underline{u},\underline{v},\underline{w},\underline{\alpha},\underline{\beta},\underline{\delta}\right)$ and $\left(\bar{u},\bar{v},\bar{w},\bar{\alpha},\bar{\beta},\bar{\delta}\right)$: Lower and Upper limits of $u, v, w, \alpha, \beta, \delta$;

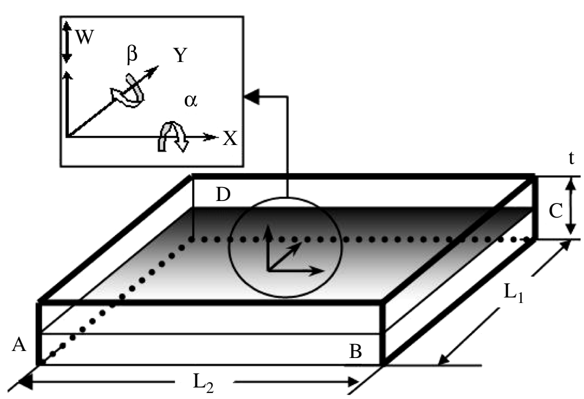
In this work, the SDT ***Small Displacement torsor with interval*** scheme has been adopted for representation of deviation of futures and ***Jacobian matrix*** has proposed to map all SDT in dimensional chain. The following sections will describe these elements.

3.1.1. Small displacement torsor with interval

The concept of the small displacement Torsor (SDT) has been developed in the seventies by P. Bourdet and A. Clément, in order to solve the general problem of the fit of a geometrical surface model to a set of points. This concept, in its first form, has largely been used in the scope of metrology. In tolerancing we are rather interested in the variations of the surface or its feature (axis, center, plane) from the nominal position. The components of these

Table 1 Small Displacement torsor with interval

ZONE PLANAR



Constraints	Torsor
Positional constraints $-\frac{t}{2} \le w \le +\frac{t}{2}$ Angular constraints $-\frac{t}{L_1} \le \alpha \le +\frac{t}{L_1}$ $-\frac{t}{L_2} \le \beta \le +\frac{t}{L_2}$	$\begin{bmatrix} [\underline{u},\bar{u}] \\ [\underline{v},\bar{v}] \\ [\underline{w},\bar{w}] \\ [\underline{\alpha},\bar{\alpha}] \\ [\underline{\beta},\bar{\beta}] \\ [\underline{\delta},\bar{\delta}] \end{bmatrix}_{FEi} = \begin{bmatrix} [0,0] \\ [0,0] \\ \left[-\frac{t}{2},+\frac{t}{2}\right] \\ \left[-\frac{t}{L_1},+\frac{t}{L_1}\right] \\ \left[-\frac{t}{L_2},+\frac{t}{L_2}\right] \\ [0,0] \end{bmatrix}_{FEi}$

variations can be represented using those of the screw parameter. Such a screw parameter is then called small displacement screw. The small displacement screw can be used directly in its generic form to represent potential variations along and about all three Cartesian axis.

In [15, 16] an inventory of all standard tolerance zones, along with their corresponding torsor representation and geometrical constraints. The torsors of a functional element presents its various possible dispersions in translation (u, v, w) and in rotation (α, β and δ) as opposed to its remaining degrees of freedom (represented here by zeros). Table 1 shows the various classes of tolerance zones with their torsor and their corresponding constraints as suggested by Desrochers and adapted, with minor changes, from [11].

Jacobian matrix: The purpose of the Jacobian matrix is to express the relation between the small displacement of all functional elements (FE) and the functional requirement (FR) sought. In this matrix therefore the columns are extracted from the various homogeneous transform matrices T_0^i relating the functional element (FE) reference frames to that of the functional requirement (FR).

$$T_0^i = \begin{bmatrix} \left[R_{3\times3}\right]_0^i & \vdots & \left[P_{3\times1}\right]_0^i \\ \cdots & \vdots & \cdots \\ 0 & \vdots & 1 \end{bmatrix} = \begin{bmatrix} \vec{C}_{1i} & \vec{C}_{2i} & \vec{C}_{3i} & \vec{d}_i \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times4} \tag{2}$$

where

$R_0^i = \left[\vec{C}_{1i}\ \vec{C}_{2i}\ \vec{C}_{3i}\right]$: These vectors represent the orientation of reference frame i with respect to 0, where the columns $\vec{C}_{1i}$, $\vec{C}_{2i}$ and $\vec{C}_{3i}$ respectively indicate the unit vectors along the axes X_i, Y_i, and Z_i of reference mark i in reference mark 0.

$\vec{d}_i = \left[dx_i\ dy_i\ dz_i\right]^T$: Position vector defining the position of the origin of reference frame *i* in 0.

The matrix jacobian is formulated by :

$$\left[J_1 \cdots J_6\right]_{FEi} = \begin{bmatrix} \left[R_0^i\right]_{3\times3} & \vdots & \left[W_i^n\right]_{3\times3} \bullet \left[R_0^i\right]_{3\times3} \\ \cdots\cdots\cdots & \vdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \left[0\right]_{3\times3} & \vdots & \left[R_0^i\right]_{3\times3} \end{bmatrix}_{6\times6} \tag{3}$$

where $\left[W_i^n\right]_{3\times3}$ is a Skew-symmetric matrix allowing the representation of the vector $\left[\vec{d}_n - \vec{d}_i\right]$ with $dx_i^n = dx_n - dx_i$, $dy_i^n = dy_n - dy_i$ and $dz_i^n = dz_n - dz_i$ knowing that $\vec{d}_n$ and $\vec{d}_i$ can be obtained from the transformation matrix in equation (2). $\left[W_i^n\right]_{3\times3} \bullet \left[R_0^i\right]_{3\times3}$ should be used to directly obtain the first three elements in the fourth, fifth and sixth columns of the Jacobian matrix. $\left[R_0^i\right]_{3\times3}$ represent the orientation matrix of reference frame *i* relative to 0 (from equation (2)). The Jacobian-torsor model can be expressed as follows:

$$\left[FR\right] = \left[J\right]\left[FE_i\right] \tag{4}$$

where
$[FR]$: 6×1 small displacements torsor of the functional requirement;
$[J]$: $6 \times n$ Jacobian matrix; with n the total number of functional elements in the chain
$[FE_i]$: 6×1 individual small displacements torsors of each part in the chain, $i = 1$ to n;

As shown, column matrix [FR] represents the dispersions around a given functional condition where the six small displacements are bounded by interval values. Similarly, the corresponding column matrix [FEs] represents the various functional elements encountered in the tolerance chain where intervals are again used to represent variations on each element. Naturally, the terms in this expression remain the same as those used in "conventional" Jacobian modeling [15,16].

3.2. UNCERTAIN MEASUREMENTS [18,19]

In this section, we introduce uncertain measurements into experimental process. For a given apparatus, we assume that we can measure a series of data x_i, and we know that these values are uncertain. Through a process of physical reasoning, testing, repeated measurements, or manufacturer's specifications, we can estimate the uncertainty magnitudes are δx_i. The values of x_i are represented by:

$$x_i = x_{im} \pm \delta x_i \tag{5}$$

where:
x_i : Desired value; and δx_i represents uncertainty measurements for x_i;
x_{im} : The best estimation value of x_i. This is based on the precision used in the measurement process; $i = 1, 2, \ldots, N$ where N is the number of measurements.

If q is a relationship with the values x_i, each containing uncertainty values:

$$q = x_1 + x_2 + x_3 + \cdots + x_N \tag{6}$$

where:
x_i : Desired value from Equation (5);
q : A composite expression from the data x_i

Because q is an expression of series of data and each have an uncertain measurement, then q have an uncertain noted δq. The expression of q is:

$$q = q_m \pm \delta q \tag{7}$$

where:
q : Desired value of the relationship with x_i;
q_m : The best estimation value calculated for q;
δq : Uncertainty measurement for q;

To obtain the parameters q, which can be used to spread the uncertainty concepts, they are divided into two possibilities: uncertainties are dependent and uncertainties are independent.

In the dependent case:

$$\delta q = \delta x_1 + \delta x_2 + \delta x_3 + \cdots + \delta x_N \tag{8}$$

In the independent case:

$$\delta q = \sqrt{(\delta x_1)^2 + (\delta x_2)^2 + \cdots + (\delta x_N)^2} \tag{9}$$

The value of q_m, is the same in both cases (dependent and independent):

$$q_m = x_{1m} + x_{2m} + x_{3m} + \cdots + x_{Nm} \tag{10}$$

Another relationship is used to statistically determine our functional requirement. When uncertainty is multiplied by a constant B, we obtain the following expressions:

$$x = x_m \pm \delta x \tag{11a}$$

$$q = B \times x \tag{11b}$$

$$q = q_m \pm \delta q \tag{11c}$$

$$\delta q = |B| \times \delta x \tag{11d}$$

These three expressions are used later in this paper to obtain the cumulative tolerance for a given functional requirement.

Note that these formulations (Equations 5 to 11) are used only for each unique parameter x_i. When a series of data for each x_i is used, these expressions will have different versions. For this version we provided statistical uncertain propagations [19].

In this section the basic principle used when studying uncertainty was that a unique value exists for each x_i. So, in order to generalize our statistical approach, a set of values x_i was obtained using Monte Carlo simulation. Thus with this statistical concept and within the context of an expression containing several uncertainties, the following general expression was obtained [19]:

$$x_i = x_{im} \pm \frac{\sigma_{\bar{x}_i}}{\sqrt{N}} \tag{12}$$

where:

- x_i : Desired value for a set of data.
- x_{im} : The best estimated value, based on the precision used in uncertain measurement processes. This value can be obtained by using mean (average of set data for x).
- $\frac{\sigma_{\bar{x}_i}}{\sqrt{N}}$: Uncertain standard deviation. This present δx_i when we use a set of data. N = Number of data used.

In summary, the x_i expressions were simulated by each element in the torsor (last column in Equation (1)). The constant B from Equation (9) became the Jacobian values. These x_{im} values are in fact averages obtained by simulations for each interval δx_i.

Clearly, if statistical tolerances are specified for the inputs (set of $(x_1, x_2, \ldots, x_N)$), a statistical tolerance can be calculated for the output (set of $(q_1, q_2, \ldots, q_N)$). This amounts to determining the average and standard deviation of the output. Details can be found in Taylor [18,19] in these approaches have a variety of names including statistical tolerance analysis, propagation of errors and variation transmission analysis. The next section shows the simulation principle adopted.

4. STATISTICAL ANALYSIS

The basic principle in simulation is to use the torsor terminal elements in order to obtain a series of random data for each torsor row (random generation of real values for each SDTI). The random variables obtained are based on the assumption that the distribution follows the normal

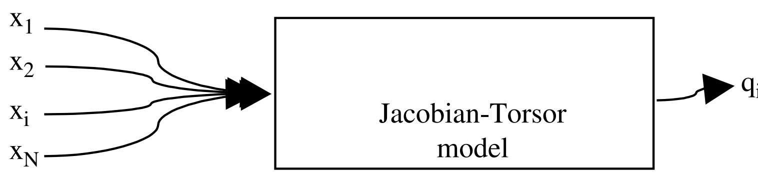


Figure 1 Monte Carlo simulation.

low. To compute a set of real values for FR, the simulation process uses the average of each series to fill the torsor functional elements. Finally we multiply the Jacobian torsor averages and the matrix. In this way we obtain torsor elements for the functional condition. Figure 1 shows the different blocks obtained and the various interactions between them:

Finally, the following relation is obtained:

$$\begin{bmatrix} u_m \\ v_m \\ w_m \\ \alpha_m \\ \beta_m \\ \delta_m \end{bmatrix}_{FR} = \left[\begin{bmatrix} J_1 J_2 J_3 J_4 J_5 J_6 \end{bmatrix}_{FE1} \ldots\ldots \begin{bmatrix} J_1 J_2 J_3 J_4 J_5 J_6 \end{bmatrix}_{FEN} \right] \cdot \begin{bmatrix} \begin{bmatrix} u_m \\ v_m \\ w_m \\ \alpha_m \\ \beta_m \\ \delta_m \end{bmatrix}_{FE1} \\ \vdots \\ \begin{bmatrix} u_m \\ v_m \\ w_m \\ \alpha_m \\ \beta_m \\ \delta_m \end{bmatrix}_{FEN} \end{bmatrix} \tag{13}$$

where:

$$[FR] = \begin{bmatrix} u_m \\ \vdots \\ \delta_m \end{bmatrix}_{FR}$$

and $[FEi] = \begin{bmatrix} u_m \\ \vdots \\ \delta_m \end{bmatrix}_{FEi}$: The same definition as in Equation (1), but in this formulation the torsors consists of data simulation averages. So, each element have a real value (not interval);

$\begin{bmatrix} J_1 \cdots J_6 \end{bmatrix}_{FEi}$: The same definition as in Equation (1).

$u_m, v_m, w_m, \alpha_m, \beta_m, \delta_m$: Best estimation of $u, v, w, \alpha, \beta, \delta$; with respect to the Lower and Upper limits. From this we obtain the average of a data series. Respected generated random value using the limits $\underline{u}, \underline{v}, \underline{w}, \underline{\alpha}, \underline{\beta}, \underline{\delta}$ and $\overline{u}, \overline{v}, \overline{w}, \overline{\alpha}, \overline{\beta}, \overline{\delta}$.

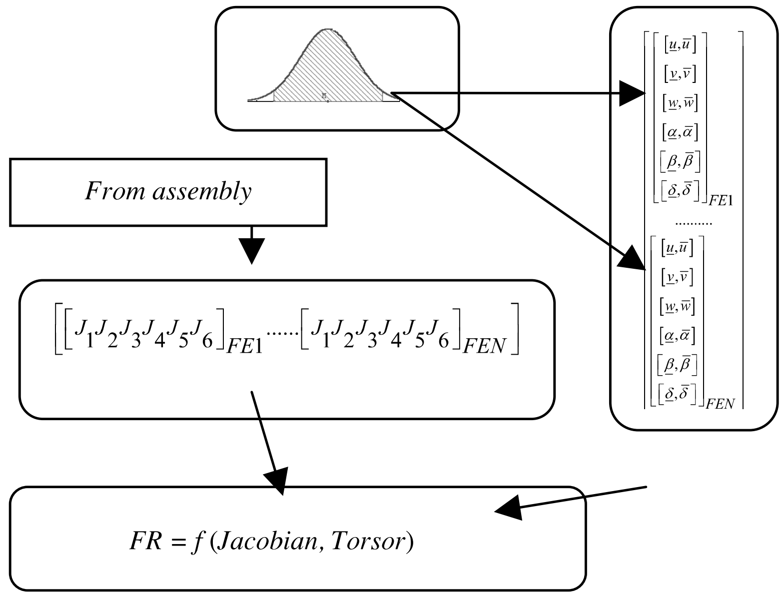


Figure 2 Methodology applied in statistical tolerance analysis using the unified model.

Using the expression (7), the functional requirement should be represented by:

$$FR_D = FR_m \pm \delta FR_D \tag{14}$$

where:

FR_D : Represents the functional requirement in direction *D*, with *D* is *x*, *y* or *z*. In this paper we concentred only on the direction in translation, and did not evaluate the rotational direction;

FR_m : Average FR or the best value of *FR* (equation (10));

δFR_D : Variation of *FR*, this variation obtains using equation (8) or (9). Each standard deviation for set of data in the torsor has to convert to standard deviation of average (using equation (12)).

The calculation method concern the parameters of the FR can be done using the expression 8 and 12. Remember, these expressions, based on the assumption that the distribution follows the normal low.

The goal of statistical tolerance analysis is to obtain the statistical limits for the FR in the direction of analysis. More specifically, the statistical tolerance analysis method consists of the flowing steps:

- **Step 1: Dimensional chain and Jacobian-Torsor model:** Based on the modeling of an assembly, we build the Jacobian-Torsor model, which is based on the principles of functional graph elements and contacts between the elements. At the end of this stage, we have a combined model, the same as the basic Equation (1).
- **Step 2: Generate N real random values within each SDTI component:** First, we provide a statistical distribution for each *FE* (in our approach the normal). Second we generate real random values within the bounds of the intervals defined by each SDTI.

- **Step 3: Average and standard deviation for torsors:** Using data from second step, we compute the average and standard deviation for each value in the torsor.
- **Step 4: Obtain the FR in direction analysis:** We apply Equation (14) model in order to obtain FR_m (note that the torsors obtain contain three directions for translation, and three directions for rotation). In this paper we use only translation directions (note that in order to multiply the Jacobian by the torsor, we select only the line corresponding to the analysis direction).

These steps represent a comprehensive method for performing statistical tolerance analysis using the Jacobian–Torsor technique. The next section shows a numerical application.

5. NUMERICAL EXAMPLES

Two numerical examples of different assemblies described in this section. These are given here to clarify the method statistical analysis. The first example demonstrates with a simple assembly the four step of our method. The second one give illustrates the effect of orientation tolerance in statistical method.

5.1. FIRST NUMERICAL EXAMPLE

The following example is basic enough to allow comparison of the computational results with those of traditional approaches. It does however include a feature that allows complete validation of the proposed approach. In the following section we show how the four steps of the tolerance analysis statistical methodology can be successfully applied to a simple three-dimensional example.

The proposed assembly is shown in Figures 3 to 5. It features three parts in perfect contact on an inclined plane, with their functional condition to be computed shown on the leftmost

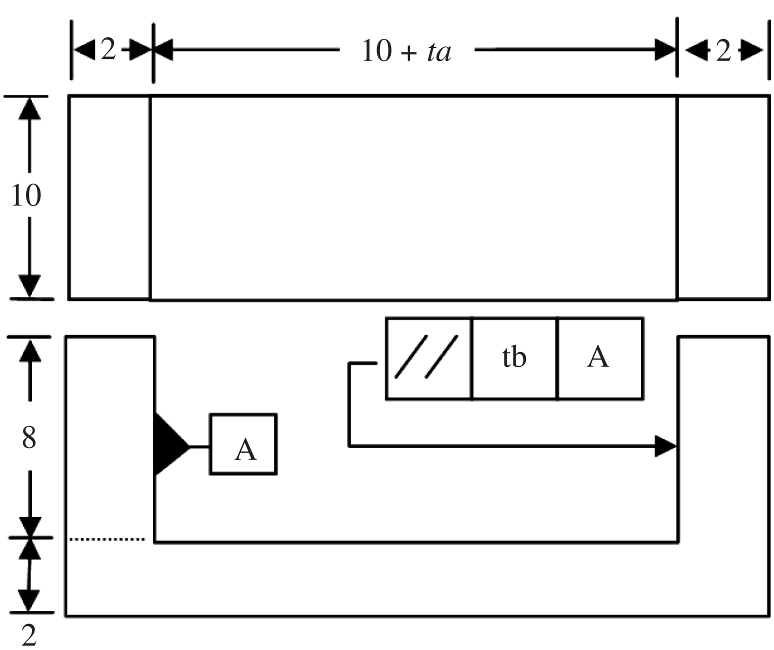


Figure 3 Definition of the concave part (part #1).

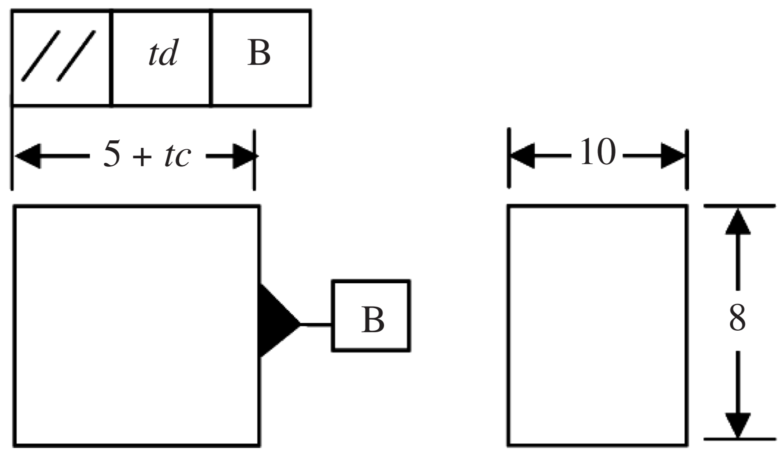


Figure 4 Definition of the convex parts (part #2, 3).

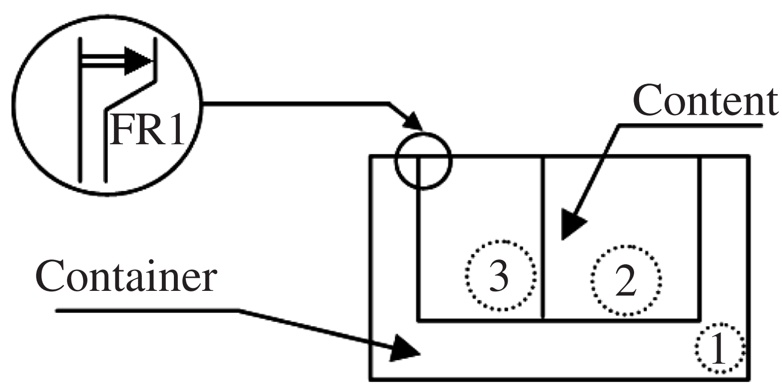


Figure 5 Definition of the functional requirement FR1.

side of the assembly. In these figures, we labelled some key tolerances from *ta* to *td*, with one functional condition as shown in Figures 5. The purpose of this example is to explore the functional condition FR1 (Figure 6).

For this simple example, the relevant parameters are as follows:

$ta = 0.1/0.3$, $tb = 0.1$, $tc = -0.2/0.00$, $td = 0.1$.

In Figure 6, contact surfaces between the parts are first identified. From this a connection graph can be constructed in order to establish the dimensional chain around this mechanism's functional condition FR1. In this example, the designer wanted to see the FR1.

Step 1: Dimensional chain and Jacobian-Torsor model: Prior to applying the principles set forth in the unified model, a dimensional chain is needed. This chain can be identified using a connection graph. The kinematic chain obtained contains three internal pairs (FE0, FE1), (FE2, FE3), (FE4, FE5) as well as two kinematic pair (FE1, FE2) and (FE3, FE4). In this example, we assumed that the reference frames are in the middle of the tolerance range or contact uncertainty zone and that they are associated with the second element in the above-defined pairs. The two kinematic torsors for (FE1, FE2) and (FE3, FE4) are considered null because the contact between the two planes is assumed perfect, and the form tolerances are not being considered here.

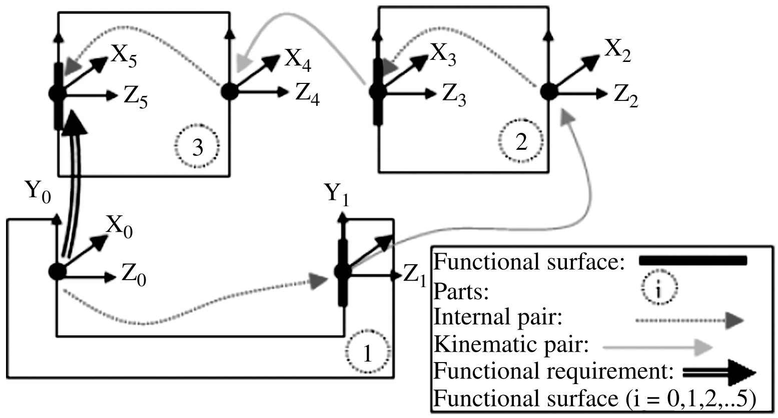


Figure 6 Kinematic chains identification.

Based on the approach Jacobian-Torsor model we obtain the expression of our model:

$$
\begin{bmatrix}[\underline{u},\overline{u}]\\ [\underline{v},\overline{v}]\\ [\underline{w},\overline{w}]\\ [\underline{\alpha},\overline{\alpha}]\\ [\underline{\beta},\overline{\beta}]\\ [\underline{\delta},\overline{\delta}]\end{bmatrix}_{FR}
=
\left[
\begin{bmatrix}1&0&0&0&10&0\\0&1&0&-10&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}_{FE1}
\begin{bmatrix}1&0&0&0&5&0\\0&1&0&-5&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}_{FE2}
\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}_{FE3}
\right]
\cdot
\begin{bmatrix}
\begin{bmatrix}[0,0]\\ [0,0]\\ [+0.1,+0.3]\\ [\pm 0.0125]\\ [\pm 0.01]\\ [0,0]\end{bmatrix}_{FE1}\\
\begin{bmatrix}[0,0]\\ [0,0]\\ [-0.2,0.0]\\ [\pm 0.01]\\ [\pm 0.0125]\\ [0,0]\end{bmatrix}_{FE2}\\
\begin{bmatrix}[0,0]\\ [0,0]\\ [-0.2,0.0]\\ [\pm 0.01]\\ [\pm 0.0125]\\ [0,0]\end{bmatrix}_{FE3}
\end{bmatrix}
\quad (15)
$$

This model forms the output for this step.

Step 2: Generate N real random values within each SDTI component: In the model of the expression (15), we can extract the limits of each parameter in order to generate random values. These values could be obtained because the distribution is normal for all elements in the left column (functional elements). Using the function Alea() or Rand(), the generation of N = 1000 values was obtained using Microsoft Excel. The following table shows the components of this simulation.

Table 2 Simulation parameters used to generate within each SDTI component

	Simulation variables			After simulation	
FE	**Parameters**		**Values**	**Average**	**Std. Dev.**
1	w1	$\left[\underline{w}, \overline{w}\right]$	[+0.1, +0.3]	0,20	0.06
2	w2		[–0.2, 0.0]	–0.10	0.06
3	w2			–0.10	0.06

Step 3: Torsors averages and standard deviations: Using the parameters from Jacobian-Torsor model (Equation (15)) and the data series from Step 2, we obtain the statistical parameters value shown in the last two columns of Table 2 (average and standard deviation).

Step 4: FR obtained from direction analysis: The functional requirement direction coincides with the axis Z_0. We therefore multiply the third row of the Jacobian matrix by the torsor averages (obtained in previous step) to obtain the FR torsor averages in the Z direction. To obtain the expression 14, we use[1] the Equation (10) and last two columns of Table 2 (average and standard deviation).

$$FR_m = (0.2) + (-0.1) + (-0.1) = 0.00 \tag{16}$$

The results represent the value of FR_m in Expression (14).

The last step consists of obtaining the dispersion δFR, (we assume the dependent case) obtained through applying Expression (12) in equation (8):

$$\sigma_{FR_D} = \delta w_1 + \delta w_2 + \delta w_3 = \frac{0.006}{\sqrt{1000}} + \frac{0.006}{\sqrt{1000}} + \frac{0.006}{\sqrt{1000}} = 0.0056 \tag{17}$$

The results represent the value of δFR_D in Expression (14). Finally, the functional requirement obtained using the statistical method is:

$$FR_D = 0.0000 \pm 0.0056 \text{ mm} \tag{18}$$

Same result (error counted is 0.23%) can be obtained when we use Monte Carlo simulation classical. This method consists to apply equation (1) in each generation random generation value. This approach takes more execution time because at each generation have to multiply the matrix Jacobian with a column vector functional element.

From the figure 7, *FR* has a normally distribution: In probability theory, if $C = A + B$, if A and B are independent random variables and identically distributed random variables that are normally distributed, then C is also normally distributed [20].

On the other hand, by obtaining the functional requirement using a classical approach we obtained the FR [0.1,0.7]*mm* we can see that the statistical values are smaller than those of the worst-case method. Actually we intend to develop a module to validate this point.

Consequently, according to the torsor $[FR]_{FR}$ for the functional condition under scrutiny, we are led to conclude that the functional condition or allowable clearance or gap in the Z_0 direction must lie within an interval of [–0.006, +0.006].

[1]We assume that the values in torsor are dependent.

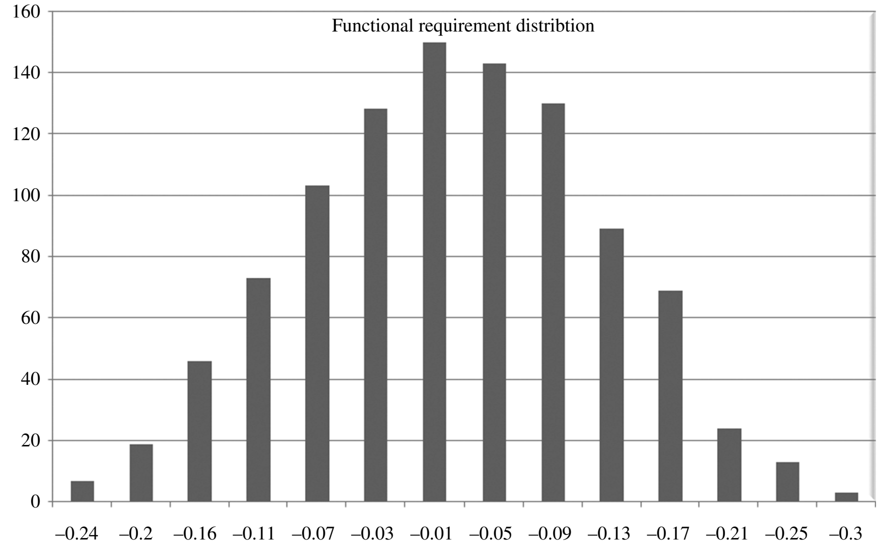


Figure 7 Output of functional requirement statistical histogram.

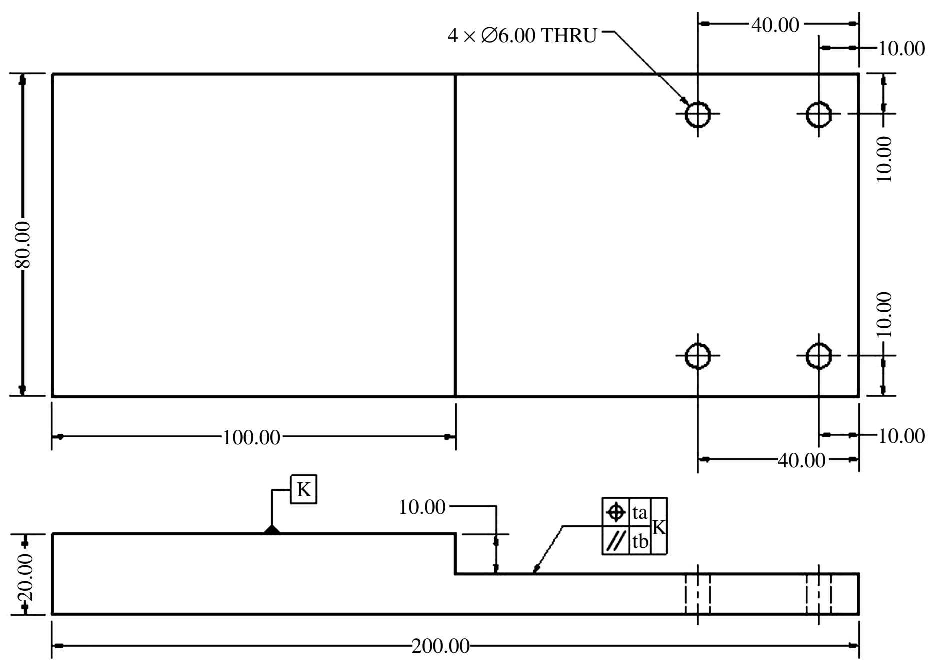


Figure 8 Base for centering pin mechanism.

In this example, we use direction analysis in our calculation and then one-third line of the line. We can see the rotational effect was not considered and this in turn affected the orientation tolerance, but this is a problem to be given further consideration in further research (see second numerical example).

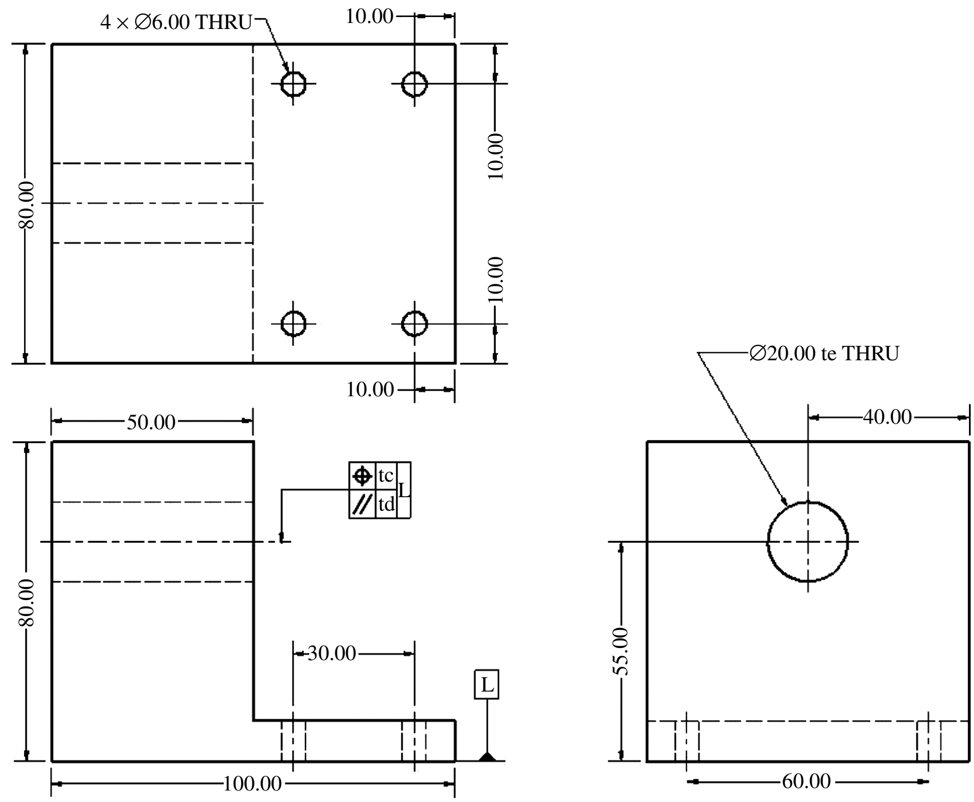


Figure 9 Block for centering pin mechanism.

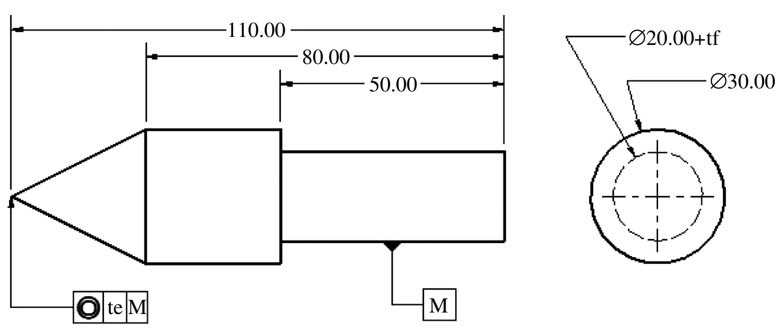


Figure 10 Pin for centering pin mechanism.

We can conclude that the tolerances imposed (in Table 2) may be expanded to ensure that manufacturing costs will be cheaper, even when we work with the deterministic method the results obtained for the condition are very close to the designer's limits. The statistical method enables us to enhance tolerance, but it does not provide any indication of the maximum expansion possible.

5.2. SECOND NUMERICAL EXAMPLE

The second example is *centering pin mechanism*. the parts of this assembly presented in Figures 8 to 11. In these figures, we labelled some key tolerances from *ta* to *te*. The

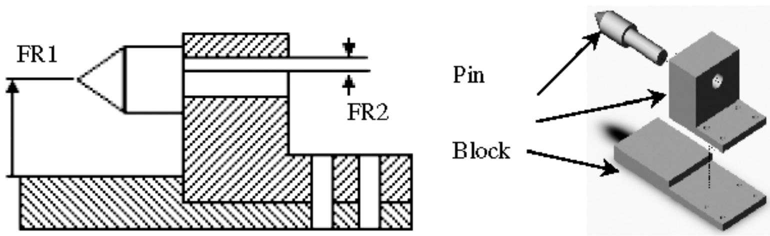


Figure 11 Detail assembly for centering pin mechanism with two FRs.

Table 3 Assigned tolerances values

Tolerances values proposed							
ta	0.2	*tc*	0.2	*te*	*H11* : 0.00/0.13	*tg*	0.1
tb	0.1	*td*	0.1	*tf*	*h8* : –0.033/0.000		

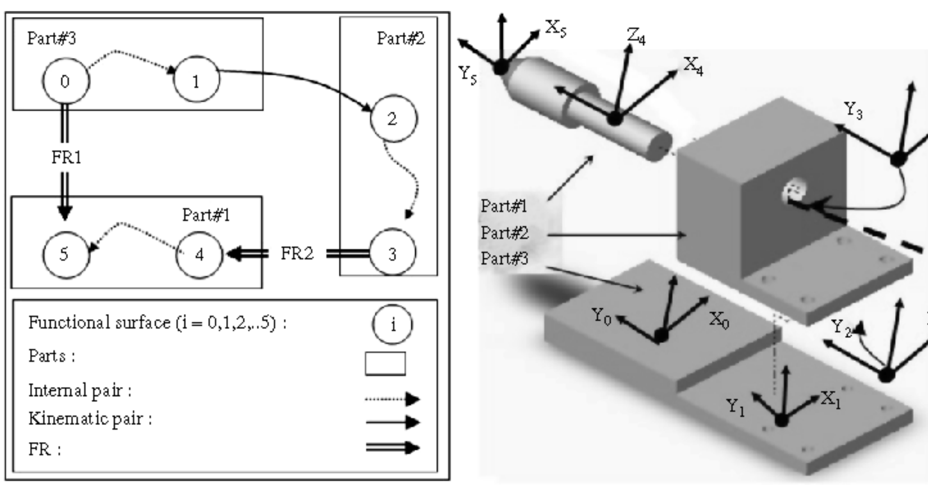


Figure 12 Kinematic chain identification.

mechanism featured three parts, with two functional conditions as shown in Figure 11. The purpose of this example is to explore the functional condition FR1.

The figure below shows the assembly drawing.

In this example, the designer specified an FR1 of ±0.5 mm. (For more information, see the analysis for this example described in [14,15]). To meet this objective the designer proposed a list of tolerances from *ta* to *tg*, as described in the table below. For this simple example, the relevant parameters are as follows:

As shown in Figure 12, the contact surfaces between the parts are first identified. A connection graph is then constructed for this mechanism in order to establish the dimensional chain around the functional condition *FR1*.

The resulting kinematic chain contains three internal pairs (FE0, FE1), (FE2, FE3), (FE4, FE5) as well as one kinematic pair (FE1, FE2). Note that there are two FRs: FR1 applies

Table 4 Torsors details

FE#	Constraints	Torsors form	Torsors values
FE1	$w=\pm(ta-tb)/2$ $\alpha=\pm ta/100$ $\beta=\pm ta/80$	$\begin{bmatrix}[0,0]\\ [0,0]\\ [\underline{w},\overline{w}]\\ [\underline{\alpha},\overline{\alpha}]\\ [\underline{\beta},\overline{\beta}]\\ [0,0]\end{bmatrix}_{FE1}$	$\begin{bmatrix}[0,0]\\ [0,0]\\ [\pm 0.1]\\ [\pm 0.003]\\ [\pm 0.0025]\\ [0,0]\end{bmatrix}_{FE1}$
FE2	$u=\pm(tc-td)/2$ $w=\pm(tc-td)/2$ $\alpha=\pm tc/50$ $\delta=\pm tc/50$	$\begin{bmatrix}[\underline{u},\overline{u}]\\ [0,0]\\ [\underline{w},\overline{w}]\\ [\underline{\alpha},\overline{\alpha}]\\ [0,0]\\ [\underline{\delta},\overline{\delta}]\end{bmatrix}_{FE2}$	$\begin{bmatrix}[\pm 0.05]\\ [0,0]\\ [\pm 0.05]\\ [\pm 0.004]\\ [0,0]\\ [\pm 0.004]\end{bmatrix}_{FE2}$
FE3	$t=ES-ei$ $u=\pm t/2$ $w=\pm t/2$ $\alpha=\pm t/50$ $\delta=\pm t/50$	$\begin{bmatrix}[\underline{u},\overline{u}]\\ [0,0]\\ [\underline{w},\overline{w}]\\ [\underline{\alpha},\overline{\alpha}]\\ [0,0]\\ [\underline{\delta},\overline{\delta}]\end{bmatrix}_{FE3}$	$\begin{bmatrix}[\pm 0.065]\\ [0,0]\\ [\pm 0.065]\\ [\pm 0.0026]\\ [0,0]\\ [\pm 0.0026]\end{bmatrix}_{FE3}$
FE4	$u=\pm tg/2$ $w=\pm tg/2$ $\alpha=\pm tg/30$ $\delta=\pm tg/30$	$\begin{bmatrix}[\underline{u},\overline{u}]\\ [0,0]\\ [\underline{w},\overline{w}]\\ [\underline{\alpha},\overline{\alpha}]\\ [0,0]\\ [\underline{\delta},\overline{\delta}]\end{bmatrix}_{FE4}$	$\begin{bmatrix}[\pm 0.05]\\ [0,0]\\ [\pm 0.05]\\ [\pm 0.0033]\\ [0,0]\\ [\pm 0.0033]\end{bmatrix}_{FE4}$

FE3 : contact element between elements 3 and 4.

between (FE0, FE5) and FR2 between (FE3, FE4) and that defined for functional fit 20 is H11/h8.

The torsor calculation method accounts for the effect that might result when a tolerance dimension or position is imposed simultaneously on the same surface. The table 4 lists the torsors and details that will be included in the Jacobian-Torsor model. From this, the final expression for the Jacobian-Torsor model and its intervals becomes:

$$\begin{bmatrix}[\underline{u},\bar{u}]\\ [\underline{v},\bar{v}]\\ [\underline{w},\bar{w}]\\ [\underline{\alpha},\bar{\alpha}]\\ [\underline{\beta},\bar{\beta}]\\ [\underline{\delta},\bar{\delta}]\end{bmatrix}_{FRr} = \left[\begin{bmatrix}1&0&0&0&-55&100\\0&1&0&55&0&0\\0&0&1&-100&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}_{FE1} \begin{bmatrix}1&0&0&0&0&75\\0&1&0&0&0&0\\0&0&1&-75&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}_{FE2} \begin{bmatrix}1&0&0&0&0&75\\0&1&0&0&0&0\\0&0&1&-75&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}_{FE3} \begin{bmatrix}1&0&0&0&45&0\\0&1&0&-45&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}_{FE4} \right] \begin{bmatrix} \begin{bmatrix}[0,0]\\ [0,0]\\ [\pm0.05]\\ [\pm0.002]\\ [\pm0.0025]\\ [0,0]\end{bmatrix}_{FE1}\\ \begin{bmatrix}[\pm0.05]\\ [0,0]\\ [\pm0.05]\\ [\pm0.004]\\ [0,0]\\ [\pm0.004]\end{bmatrix}_{FE2}\\ \begin{bmatrix}[\pm0.0815]\\ [0,0]\\ [\pm0.0815]\\ [\pm0.0033]\\ [0,0]\\ [\pm0.0033]\end{bmatrix}_{FE3}\\ \begin{bmatrix}[\pm0.05]\\ [0,0]\\ [\pm0.05]\\ [\pm0.0033]\\ [0,0]\\ [\pm0.0033]\end{bmatrix}_{FE4} \end{bmatrix} \tag{19}$$

The torsor components corresponding to undetermined elements in the kinematic pair can then be replaced by null elements, because they do not affect small displacements along the analysis direction (*w* in this case, on axis Z_0). Thus, the calculated functional condition becomes.

In the statistical approach, the random variables generated using last column (Equation (7)). Following this generation and based on the principle described in Section 3, the dispersion based on this condition becomes functional (where *D* represents the direction along the axe *Z*):

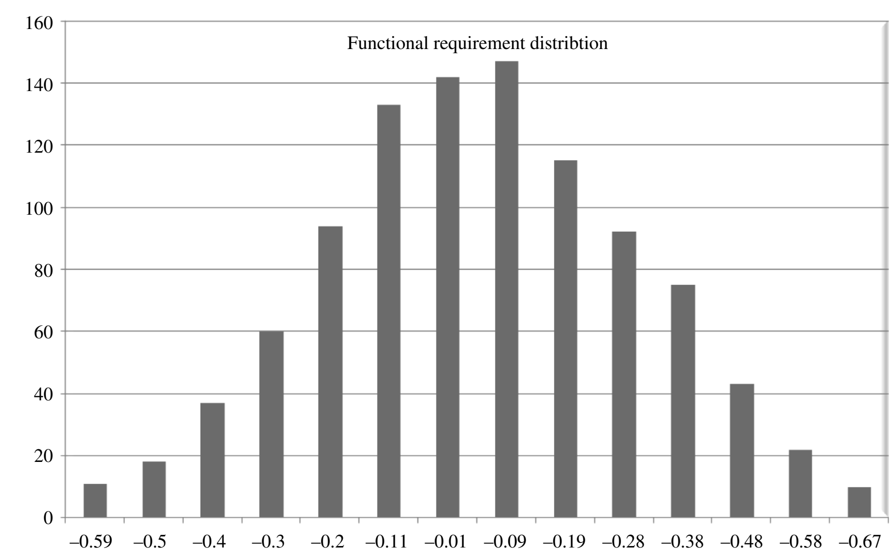


Figure 13 Functional requirement distribution.

$$FR_D = FR_{Dm} \pm \delta FR_D = FR_Z \ = \ 0.00 \ \pm \ 0.27 \text{ mm} \tag{20}$$

We can then see that the statistical dispersion is smaller than that of the deterministic method because: The deterministic method assumes that mistakes happen all at once and are most probable. By contrast, the statistical method assumes that the maximum values are generated (distribution is normally low) around the average values (in our case the average is zero). By using the approach described in [15,16], we obtain the following deterministic method: Along the Z direction, $FR_Z = \pm 0.976$ mm. We can conclude that the tolerances imposed (Table 3) may be expanded to ensure that manufacturing costs will be cheaper, even when we work with the deterministic method the results obtained for the condition are very close to the designer's limits. The statistical method enables us to enhance tolerance, but it does not provide any indication of the maximum expansion possible. Remark: in this example, we see that the orientations of the second and third torsor are considered in Jacobian-torsor model.

6. CONCLUSION

Statistical tolerancing has been the focus of extensive research for many years, due to its importance in design and manufacturing. This paper presented a new tolerance modeling technique based on the combination of two previously distinct approaches: the Jacobian model and the torsor using the measurement uncertainty propagation theory. In addition to this combination, the proposed modeling technique also introduced interval arithmetic, therefore allowing tolerance values or constraints to be directly embedded in the torsor or screw formulation. This in turn made it possible to calculate tolerances based computations on a "tolerance zone basis" rather than on a "point basis," consequently avoiding costly simulation using points extracted from the boundaries of the various surfaces involved along a tolerancing chain. This interval then became the constraints used in the generation of random values for each element in every torsor.

- In Jacobian-Torsor, according to probability theory, *FR* has a normally distribution. When all parameters in torsor are identically distributed random variables that are normally distributed.
- Future work: in this present work we used only normal distribution; and we intend to extend our approach by considering other distribution probabilities. In this paper, we explored the dependence measurement value. In future work however we intend to provide more details on measurement independence.
- Finally, this method can be applied using CAD systems and the geometrical specifications defined by standards.

ACKNOWLEDGEMENTS

The authors would like to acknowledge the Université du Québec en Abitibi-Témiscaming **(UQAT)** for their financial contribution to this ongoing project.

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