

On Preserver Mappings of Spectrum and Spectral Radius of Linear Operators in Banach Spaces

Iman Bahreyni¹, Rahmat Soltani², Fariba Ershad³

¹Mathematics Major, Department of Mathematics, Payame Noor University, Tehran, Iran.

²Assistant Professor, Department of Mathematics, Payame Noor University, Tehran, Iran.

³Assistant Professor, Department of Mathematics, Payame Noor University, Tehran, Iran.

*Corresponding author

Abstract

The purpose of this paper is to present some results related to the theory of pseudospectra. Given a complex Banach space X , we denote by $\mathcal{L}(X)$ the algebra of all linear bounded operators on it. For $\varepsilon > 0$, let $r_\varepsilon(T)$ denote the ε -pseudospectral radius of $T \in \mathcal{L}(X)$ and denote by I the identity operator on X . For $T \in \mathcal{L}(X)$, by $\sigma(T)$ we shall denote its spectrum and by $r(T)$ its spectral radius. For an infinite-dimensional space X , we characterize surjective maps $\varphi: \mathcal{L}(X) \rightarrow \mathcal{L}(X)$ such that $r_\varepsilon(\varphi(T_1)\varphi(T_2)) = r_\varepsilon(T_1T_2)$ for all $T_1, T_2 \in \mathcal{L}(X)$. An analogous result is obtained in the finite-dimensional case, with no surjectivity assumption on the map φ .

Keywords: Nonlinear Preserver Mapping, Spectrum, Pseudo Spectrum, Spectral Radius, Pseudo Spectral Radius, Banach Space

1. Introduction

Over the past few years, there has been a considerable interest in linear mappings on algebras of operators that preserve certain properties of operators. In particular, a problem how to characterize linear maps that preserve the spectrum of each operator has attracted the attention of many mathematicians. In [16], Jafarian and Sourour proved that a surjective linear spectrum-preserving map of $\mathfrak{B}(X)$ onto $\mathfrak{B}(Y)$, where X and Y are Banach spaces, is either an algebra-isomorphism or an antiisomorphism. A similar result for finite-dimensional spaces was obtained much earlier by Marcus and Moyls [17]. Recently, Aupetit and Mouton [18] extended the result of Jafarian and Sourour to primitive Banach algebras with minimal ideals.

The purpose of this paper is to present some results related to the theory of pseudospectra (see, for example, [3], [11] and the book [15]), and then to use them for the study of a nonlinear preserver problem which is stated in terms of the pseudospectral radius. Let $\mathcal{L}(X)$ be the algebra of all bounded linear operators on a complex Banach space X , and denote by I the identity operator on X . For $T \in \mathcal{L}(X)$, by $\sigma(T)$ we shall denote its spectrum and by $r(T)$ its spectral radius. For $\varepsilon > 0$, the ε -pseudospectrum of a linear and continuous operator T defined on X is usually defined as either

$$\sigma_\varepsilon(T) = \{\lambda \in \mathbb{C}: \|(\lambda I - T)^{-1}\| \geq 1/\varepsilon\} \quad (1)$$

or

$$\omega_\varepsilon(T) = \{\lambda \in \mathbb{C}: \|(\lambda I - T)^{-1}\| > 1/\varepsilon\} \quad (2)$$

with the convention that $\|(\lambda I - T)^{-1}\| = +\infty$ if $\lambda I - T \in \mathcal{L}(X)$ is not invertible. (The norm used on $\mathcal{L}(X)$ is just the standard operatorial one, that is $\|T\| = \sup\{\|Tx\|: \|x\| \leq 1\}$.) Therefore, the inclusions

$$\sigma(T) \subseteq \omega_\varepsilon(T) \subseteq \sigma_\varepsilon(T)$$

always hold, with $\omega_\varepsilon(T) \subseteq \mathbb{C}$ being open and $\sigma_\varepsilon(T) \subseteq \mathbb{C}$ being compact. The inclusion $\omega_\varepsilon(T) \subseteq \sigma_\varepsilon(T)$ is always strict, the closure of $\omega_\varepsilon(T)$ is a subset of $\sigma_\varepsilon(T)$, but the last inclusion may be strict (see, for example, [12, Theorem

3.1)): this happens because the norm of the resolvent of a linear bounded operator may be constant on an open set. Still,

$$\sup\{|\lambda|: \lambda \in \omega_\varepsilon(T)\} = \sup\{|\lambda|: \lambda \in \sigma_\varepsilon(T)\}. \quad (3)$$

To see this, we shall use the following result of Globevnik [10, Prop. 1]: if Ω is an open subset in the unbounded component of $\mathbb{C} \setminus \sigma(T)$ and $M > 0$ is such that $\|(\lambda I - T)^{-1}\| \leq M$ for each $\lambda \in \Omega$, then $\|(\lambda I - T)^{-1}\| < M$ for each $\lambda \in \Omega$. In order to obtain (3), denote $R = \sup\{|\lambda|: \lambda \in \sigma_\varepsilon(T)\}$. There exists then $\lambda_0 \in \sigma_\varepsilon(T)$ such that $R = |\lambda_0|$, and since $R > r(T)$, then λ_0 is in the unbounded component of $\mathbb{C} \setminus \sigma(T)$. If $\lambda_0 \in \omega_\varepsilon(T)$, the two suprema are equal. If not, then $\|(\lambda_0 I - T)^{-1}\| = 1/\varepsilon$. If there is no sequence $(\lambda_n)_{n \geq 1} \subseteq \omega_\varepsilon(T)$ such that $\lambda_n \rightarrow \lambda_0$, there exists $\delta > 0$ such that $|\lambda - \lambda_0| < \delta$ implies $|\lambda| > r(T)$ and $\lambda \notin \omega_\varepsilon(T)$, that is $\|(\lambda I - T)^{-1}\| \leq 1/\varepsilon$. The above stated result gives then $\|(\lambda I - T)^{-1}\| < 1/\varepsilon$ for $|\lambda - \lambda_0| < \delta$, arriving to a contradiction. Thus $|\lambda_n| \rightarrow |\lambda| = R$, for some sequence $(\lambda_n)_n \subseteq \omega_\varepsilon(T)$, which implies again that the two suprema in (3) are equal.

Motivated by its classical counterpart, the common value of the two suprema in (3) will be called the ε -pseudospectral radius of T , and shall be denoted by $r_\varepsilon(T)$. For $T \in \mathcal{L}(X)$, the following equality always holds

$$\omega_\varepsilon(T) = \bigcup_{\|E\| < \varepsilon} \sigma(T + E). \quad (4)$$

(See, for example, [3].) We also have that

$$\text{clos}(\omega_\varepsilon(T)) \subseteq \text{clos}\left(\bigcup_{\|E\| < \varepsilon} \sigma(T + E)\right) \subseteq \sigma_\varepsilon(T), \quad (5)$$

where ‘clos’ denotes the closure (see, for example, [13]). Then (3), (4) and (5) give

$$\begin{aligned} r_\varepsilon(T) &= \sup\{r(T + E): \|E\| < \varepsilon\} \\ &= \sup\{r(T + E): \|E\| \leq \varepsilon\}. \end{aligned}$$

In particular,

$$r(T) + \varepsilon \leq r_\varepsilon(T) \leq \|T\| + \varepsilon \quad (T \in \mathcal{L}(X)) \quad (6)$$

and

$$r_\varepsilon(UTU^{-1}) = r_\varepsilon(T) \quad (T \in \mathcal{L}(X)),$$

if $U \in \mathcal{L}(X)$ is a bijective isometry.

Let us also observe that, directly from the definition,

$$r_\varepsilon(\lambda T) = |\lambda| r_{\varepsilon/|\lambda|}(T) \quad (T \in \mathcal{L}(X), \lambda \in \mathbb{C} \setminus \{0\}).$$

In particular,

$$r_\varepsilon(\xi T) = r(T) \quad (T \in \mathcal{L}(X), |\xi| = 1).$$

2. Statement of the main results

The following result is proved in [2]. It extends to the case of ε -pseudospectral radius the corresponding result for the case of the ε -pseudospectrum which was obtained in [6, Theorem 4.1].

Theorem 1. ([2, Theorem 1.1]) Let H be an infinite-dimensional complex Hilbert space and let $\varepsilon > 0$. A surjective map $\varphi: \mathcal{L}(H) \rightarrow \mathcal{L}(H)$ satisfies

$$r_\varepsilon(\varphi(T_1)\varphi(T_2)) = r_\varepsilon(T_1 T_2) \quad (T_1, T_2 \in \mathcal{L}(H))$$

if and only if there exist a functional $\xi: \mathcal{L}(H) \rightarrow \mathbb{C}$ with $|\xi(T)| = 1$ for every $T \in \mathcal{L}(H)$ and a unitary or conjugate unitary operator $U: H \rightarrow H$ such that

$$\varphi(T) = \xi(T)UTU^* \quad (T \in \mathcal{L}(H)).$$

A similar result also holds in the finite dimensional case. It extends the corresponding result for the case of the ε -pseudospectrum which was obtained in [4, Theorem 3.3]. Throughout this paper, by $\mathcal{M}_n$ we shall denote the algebra of all $n \times n$ matrices over the complex field.

Theorem 2. ([2, Theorem 1.2]) Let $n \geq 3$ and let $\varepsilon > 0$. A map $\varphi: \mathcal{M}_n \rightarrow \mathcal{M}_n$ satisfies

$$r_\varepsilon(\varphi(T_1)\varphi(T_2)) = r_\varepsilon(T_1T_2) \quad (T_1, T_2 \in \mathcal{M}_n)$$

if and only if there exist a functional $\xi: \mathcal{M}_n \rightarrow \mathbb{C}$ with $|\xi(T)| = 1$ for every $T \in \mathcal{M}_n$ and a unitary matrix $U \in \mathcal{M}_n$ such that either

$$\varphi(T) = \xi(T)UTU^* \quad (T \in \mathcal{M}_n),$$

or

$$\varphi(T) = \xi(T)U\bar{T}U^* \quad (T \in \mathcal{M}_n).$$

(By $\bar{T} \in \mathcal{M}_n$ we denote the matrix obtained from $T \in \mathcal{M}_n$ by entrywise complex conjugation, and by $T^* \in \mathcal{M}_n$ the conjugate of its transpose.)

It was left as an open question in [2] under which conditions the assertions of Theorem 1 and Theorem 2 remain true in some Banach spaces. The aim of this paper is to prove that the statements remain true for every Banach space. In the infinite-dimensional case, the following result holds.

Theorem 3. Let X be an infinite-dimensional complex Banach space and let $\varepsilon > 0$. A surjective map $\varphi: \mathcal{L}(X) \rightarrow \mathcal{L}(X)$ satisfies

$$r_\varepsilon(\varphi(T_1)\varphi(T_2)) = r_\varepsilon(T_1T_2) \quad (T_1, T_2 \in \mathcal{L}(X)) \quad (7)$$

if and only if there exist a functional $\xi: \mathcal{L}(X) \rightarrow \mathbb{C}$ with $|\xi(T)| = 1$ for every $T \in \mathcal{L}(X)$ and a bijective linear or conjugate linear operator $U: X \rightarrow X$ such that $\|U(x)\| = \|x\|$ for each $x \in X$ and

$$\varphi(T) = \xi(T)UTU^{-1} \quad (T \in \mathcal{L}(X)). \quad (8)$$

The corresponding result for the finite-dimensional case is given by the next theorem. In the statement of Theorem 2, the norm on $\mathbb{C}^n$ is the Euclidian one (which turns $\mathbb{C}^n$ into a Hilbert space), while in the next

statement the norm on $\mathbb{C}^n$ is an arbitrary one.

Theorem 4. Let $n \geq 3$ and let $\varepsilon > 0$. A map $\varphi: \mathcal{M}_n \rightarrow \mathcal{M}_n$ satisfies

$$r_\varepsilon(\varphi(T_1)\varphi(T_2)) = r_\varepsilon(T_1T_2) \quad (T_1, T_2 \in \mathcal{M}_n). \quad (9)$$

if and only if there exist a functional $\xi: \mathcal{M}_n \rightarrow \mathbb{C}$ with $|\xi(T)| = 1$ for every $T \in \mathcal{M}_n$ and an invertible matrix $U \in \mathcal{M}_n$ such that either $\|U(x)\| = \|x\|$ for each $x \in \mathbb{C}^n$ and

$$\varphi(T) = \xi(T)UTU^{-1} \quad (T \in \mathcal{M}_n), \quad (10)$$

or $\|U(x)\| = \|\bar{x}\|$ for each $x \in \mathbb{C}^n$ and

$$\varphi(T) = \xi(T)U\bar{T}U^{-1} \quad (T \in \mathcal{M}_n). \quad (11)$$

3. Proofs

A basic tool for the proof of both [2, Theorem 1.1] and [2, Theorem 1.2] is [2, Lemma 2.6], stating that for a linear continuous operator T on a Hilbert space and $\varepsilon > 0$, we have that $r_\varepsilon(T) = \varepsilon$ if and only if $T = 0$. The next result says that this also holds for operators defined on a general complex Banach space X . The main idea of its proof is contained in [14]. (See also [9, Corollary 4.5].)

Theorem 5. Let $T \in \mathcal{L}(X)$ such that

$$r_\varepsilon(T) = \varepsilon.$$

Then $T = 0$.

Proof. Since $r(T) + \varepsilon \leq r_\varepsilon(T) = \varepsilon$, then $r(T) = 0$. Thus $\lambda I - T \in \mathcal{L}(X)$ is invertible for every non-zero $\lambda \in \mathbb{C}$, and therefore $I - \mu T \in \mathcal{L}(X)$ is invertible for every $\mu \in \mathbb{C}$.

Observe now that if $|\lambda| > \varepsilon$ then $\lambda \notin \omega_\varepsilon(T)$, and therefore $\|(\lambda I - T)^{-1}\| \leq 1/\varepsilon$. Passing with $|\lambda| \rightarrow \varepsilon$ we see that $\|(\lambda I - T)^{-1}\| \leq 1/\varepsilon$ for every $|\lambda| = \varepsilon$. This gives $\|(I - \mu T)^{-1}\| \leq 1$ for every $|\mu| = 1/\varepsilon$. The analytic function $\mu \rightarrow (I - \mu T)^{-1}$ is well-defined on $\mathbb{C}$, and since the norm of its value at $0 \in \mathbb{C}$ is 1, the maximum principle implies that

$$\|(I - \mu T)^{-1}\| = 1 \quad (|\mu| \leq 1/\varepsilon). \quad (12)$$

Consider now an arbitrary $r \in (0, 1/\varepsilon)$. From (12) we infer that

$$\|(\xi I - rT)^{-1}\| = 1 \quad (|\xi| = 1). \quad (13)$$

For every natural number $n \geq 1$, let $T_n = I + (rT) + \cdots + (rT)^n$. Denoting $\xi_k = e^{2\pi ki/(n+1)}$ for $k = 1, \dots, n$, then

$$T_n = \prod_{k=1}^n (rT - \xi_k I).$$

Using (13), we obtain that for every $n \geq 1$,

$$\|T_n^{-1}\| = \left\| \prod_{k=1}^n (rT - \xi_k I)^{-1} \right\| \leq \prod_{k=1}^n \|(rT - \xi_k I)^{-1}\| = 1.$$

Observe now that

$$T_n^{-1} = ((rT)^{n+1} - I)^{-1}(rT - 1) \quad (n \geq 1).$$

Since T is quasinilpotent, then $T_n^{-1} \rightarrow I - rT$. This gives $\|I - rT\| \leq 1$. Analogously one can prove that $\|I + rT\| \leq 1$. A classical fact due to Kakutani says that the identity is an extreme point of the closed unit ball of $\mathcal{L}(X)$, and therefore T must be the zero operator. $\square$

We shall also need the following observations regarding the ε -pseudospectral radius of rank-one operators. In the case of operators acting on a Hilbert space, the ε -pseudospectral radius of rank-one operators can be computed explicitly [6, Lemma 2.5].

For a nonzero $x \in X$ and a nonzero linear functional f in the dual X^* of X , by $x \otimes f \in \mathcal{L}(X)$ we denote the rank-one operator sending $y \in X$ to $f(y)x \in X$.

Lemma 6. Let $x \in X$ and $f \in X^*$, $(\varepsilon_n)_n \subseteq (0, +\infty)$, such that $\varepsilon_n \rightarrow 0$ and $(\alpha_n)_n \subseteq \mathbb{C}$.

i) If $f(x) \neq 0$ and $|\alpha_n| \rightarrow +\infty$, then

$$r_{\varepsilon_n}(\alpha_n x \otimes f) \rightarrow +\infty$$

ii) If $|\alpha_n| \rightarrow t \in [0, +\infty)$, then

$$r_{\varepsilon_n}(\alpha_n x \otimes f) \rightarrow t|f(x)|.$$

Proof. i) We have $r(x \otimes f) = |f(x)|$, and then for each n using the first inequality in (6) we have

$$r_{\varepsilon_n}(\alpha_n x \otimes f) \geq r(\alpha_n x \otimes f) + \varepsilon_n = |\alpha_n||f(x)| + \varepsilon_n.$$

Since $|\alpha_n||f(x)| + \varepsilon_n \rightarrow +\infty$, then $r_{\varepsilon_n}(\alpha_n x \otimes f) \rightarrow +\infty$.

ii) From

$$r_{\varepsilon_n}(\alpha_n x \otimes f) \geq |\alpha_n||f(x)| + \varepsilon_n \quad (n \in \mathbb{N}),$$

since $|\alpha_n||f(x)| + \varepsilon_n \rightarrow t|f(x)|$, then $\liminf r_{\varepsilon_n}(\alpha_n x \otimes f) \geq t|f(x)|$.

Write now $\alpha_n = |\alpha_n|\xi_n$ for each n , where $|\xi_n| = 1$. Let $\varepsilon > 0$. For $D = \{z \in \mathbb{C}: |z| < t|f(x)| + \varepsilon\}$, using the upper semi-continuity of the spectrum [1, Theorem 3.4.2] we find $\delta > 0$ such that $\|T\| < \delta$ implies $\sigma(tx \otimes f + T) \subseteq D$. Let $N \in \mathbb{N}$ such that $n \geq N$ implies $\varepsilon_n < \delta/2$ and $\|\alpha_n\| - t\|x \otimes f\| < \delta/2$. Then $n \geq N$ and $\|T\| \leq \varepsilon_n$ implies $\| |\alpha_n|x \otimes f + \bar{\xi}_n T - tx \otimes f \| < \delta$, which gives

$$\sigma(|\alpha_n|x \otimes f + \bar{\xi}_n T) \subseteq D.$$

We have

$$r(\alpha_n x \otimes f + T) = r(\xi_n(|\alpha_n|x \otimes f + \bar{\xi}_n T)) = r(|\alpha_n|x \otimes f + \bar{\xi}_n T),$$

and therefore

$$r(\alpha_n x \otimes f + T) \leq t|f(x)| + \varepsilon \quad (n \geq N, \|T\| \leq \varepsilon_n)$$

Taking the supremum over T , then

$$r_{\varepsilon_n}(\alpha_n x \otimes f) \leq t|f(x)| + \varepsilon \quad (n \geq N).$$

This gives $\limsup r_{\varepsilon_n}(\alpha_n x \otimes f) \leq t|f(x)| + \varepsilon$ for every $\varepsilon > 0$, and therefore $\limsup r_{\varepsilon_n}(\alpha_n x \otimes f) \leq t|f(x)|$. $\square$

The next theorem gives estimations for the ε -pseudospectral radius of square-zero operators. In the case of a Hilbert space, the ε -pseudospectral radius of those operators can be computed explicitly [6, Prop. 2.4]: if $T \in \mathcal{L}(H)$ for a Hilbert space H and $T^2 = 0$, then $r_\varepsilon(T) = \sqrt{\varepsilon^2 + \varepsilon\|T\|}$.

Theorem 7. Let $T \in \mathcal{L}(H)$ such that $T^2 = 0$. Then

$$\sqrt{\varepsilon\|T\|} \leq r_\varepsilon(T) \leq \frac{\varepsilon + \sqrt{\varepsilon^2 + 4\varepsilon\|T\|}}{2}.$$

Proof. The inequalities clearly hold for $T = 0$, so suppose for the remaining of the proof that $T \neq 0$. Since $T^2 = 0$, then one can easily check that

$$(\lambda I - T)^{-1} = I/\lambda + T/\lambda^2 \quad (\lambda \in \mathbb{C} \setminus \{0\}).$$

Let $(f_n)_n \subseteq X^*$ and $(x_n)_n \subseteq X$ such that $|f_n(Tx_n)| \rightarrow \|T\|$ with $f_n(Tx_n) \neq 0$ and $\|f_n\| = \|x_n\| = 1$ for every n . For each n , let $\xi_n \in \mathbb{C}$ be of modulus one such that

$$\begin{aligned} \left| \frac{f_n(x_n)}{\xi_n \sqrt{\varepsilon|f_n(Tx_n)|}} + \frac{f_n(Tx_n)}{\xi_n^2 \varepsilon |f_n(Tx_n)|} \right| &= \frac{|f_n(x_n)|}{\sqrt{\varepsilon|f_n(Tx_n)|}} + \frac{|f_n(Tx_n)|}{\varepsilon |f_n(Tx_n)|} \\ &= \frac{|f_n(Tx_n)|}{\sqrt{\varepsilon|f_n(Tx_n)|}} + \frac{1}{\varepsilon} \\ &\geq \frac{1}{\varepsilon}. \end{aligned}$$

For $\lambda_n := \xi_n \sqrt{\varepsilon|f_n(Tx_n)|} \in \mathbb{C}$, $n = 1, 2, \dots$, we have that $|f_n((I/\lambda_n + T/\lambda_n^2)(x_n))| \geq 1/\varepsilon$, which gives $\|I/\lambda_n + T/\lambda_n^2\| \geq 1/\varepsilon$. Thus $\lambda_n \in \sigma_\varepsilon(T)$ for each n , which gives $|\lambda_n| \leq r_\varepsilon(T)$ for every n . Letting $n \rightarrow \infty$, this gives the left hand side inequality from the statement.

Consider now a non-zero $\lambda \in \mathcal{W}_\varepsilon(T)$. Since $\|(\lambda I - T)^{-1}\| > 1/\varepsilon$, there exist $f \in X^*$ and $x \in X$, each of norm one, such that $|f(x)/\lambda + f(Tx)/\lambda^2| > 1/\varepsilon$. Then $|f(x)|/|\lambda| + |f(Tx)|/|\lambda|^2 > 1/\varepsilon$, so let us write $|f(x)|/|\lambda| + |f(Tx)|/|\lambda|^2 = 1/\gamma$ for some $0 < \gamma < \varepsilon$. Then $|\lambda|^2 - \gamma|f(x)||\lambda| - \gamma|f(Tx)| = 0$, which gives

$$|\lambda| = \frac{\gamma|f(x)| + \sqrt{\gamma^2|f(x)|^2 + 4\gamma|f(Tx)|}}{2}.$$

Now $|f(x)| \leq 1$, $|f(Tx)| \leq \|T\|$ and $\gamma < \varepsilon$, and therefore $|\lambda| \leq (\varepsilon + \sqrt{\varepsilon^2 + 4\varepsilon\|T\|})/2$. Taking the sup over $|\lambda|$, we obtain the right hand side inequality from the statement. $\square$

As a corollary, we obtain the following asymptotic behavior for the pseudospectral radius of square-zero operators.

Corollary 8. If $T^2 = 0$ in $\mathcal{L}(X)$, then

$$\lim_{\delta \rightarrow 0} \frac{r_\delta(T)}{\sqrt{\delta}} = \sqrt{\|T\|}. \quad (14)$$

3.1 .The infinite-dimensional case: proof of Theorem 3

One can easily check that maps of the form (8) satisfy (7). For the converse, the starting point is Theorem 5 and the following result which determines the structure on the set of rank-one operators of mappings preserving zero-product of operators in both directions.

Lemma 9. (See [5, Lemma 2.2].) Suppose X is an infinite-dimensional complex Banach space and let $\varphi: \mathcal{L}(X) \rightarrow \mathcal{L}(X)$ be a surjective map such that

$$\varphi(T_1)\varphi(T_2) = 0 \Leftrightarrow T_1T_2 = 0 \quad (T_1, T_2 \in \mathcal{L}(X)). \quad (15)$$

There exist then a functional $\alpha: \mathcal{L}(X) \rightarrow \mathbb{C} \setminus \{0\}$ and an invertible linear or conjugate linear bounded operator U on X such that

$$\varphi(T) = \alpha(T)UTU^{-1} \quad (T \in \mathcal{L}(X), \text{rank}(T) = 1). \quad (16)$$

So suppose now that X is an infinite-dimensional Banach space, and let $\varphi: \mathcal{L}(X) \rightarrow \mathcal{L}(X)$ be a surjective map such that (7) holds. Then Theorem 5 and Lemma 9 imply the existence of α and U as in the statement of the lemma such that (16) holds. Thus given any $x \in X$ and $f \in X^*$, in the case when U is linear we have that

$$\begin{aligned} \varphi(x \otimes f) &= \alpha(x \otimes f)U(x \otimes f)U^{-1} \\ &= \alpha(x \otimes f)(Ux) \otimes ((U^{-1})^*f). \end{aligned}$$

While in the case when $U: X \rightarrow X$ is conjugate linear we have that

$$\varphi(x \otimes f) = \alpha(x \otimes f)(Ux) \otimes \tilde{f},$$

where $\tilde{f} \in X^*$ is given by $y \rightarrow \overline{f(U^{-1}y)}$. Let us also observe that in the first case we have $((U^{-1})^*f)(Ux) = f(x)$, while in the second one we have $\tilde{f}(Ux) = \overline{f(x)}$. If we take the modulus, we obtain the same values.

Suppose now that $U: X \rightarrow X$ is linear: the remaining case, when U is conjugate linear, follows exactly in the same way. Since φ is surjective, there exists $Q \in \mathcal{L}(X)$ such that $\varphi(Q) = I$. Then (7) gives

$$r_\varepsilon(\varphi(T)) = r_\varepsilon(QT) \quad (T \in \mathcal{L}(X)). \quad (17)$$

For $T = nx \otimes f$ in (17), where $n \geq 1$ is a natural number and $x \in X, f \in X^*$, we get

$$r_{\varepsilon/n}(\alpha(nx \otimes f)(Ux) \otimes (U^{-1})^*f) = r_{\varepsilon/n}((Qx) \otimes f). \quad (18)$$

Suppose now that $f(x) \neq 0$. Then $((U^{-1})^*f)(Ux) \neq 0$, and let us prove that

$$\lim_{n \rightarrow +\infty} |\alpha(nx \otimes f)| = \frac{|f(Qx)|}{|f(x)|}. \quad (19)$$

If this is not the case, then either one can find a subsequence $(n_k)_k$ of $\mathbb{N}$ such that $|\alpha(n_k x \otimes f)| \rightarrow +\infty$, or one can find a subsequence $(n_k)_k$ of $\mathbb{N}$ such that $|\alpha(n_k x \otimes f)| \rightarrow t \neq |f(Qx)|/|f(x)|$. By Lemma 6, in the first case we have $r_{\varepsilon/n_k}(\alpha(n_k x \otimes f)(Ux) \otimes (U^{-1})^*f) \rightarrow +\infty$ and in the second one we have $r_{\varepsilon/n_k}(\alpha(n_k x \otimes f)(Ux) \otimes (U^{-1})^*f) \rightarrow t|f(x)|$. Now observe that again by Lemma 6 in both cases we have

$r_{\varepsilon/n_k}((Qx) \otimes f) \rightarrow |f(Qx)|$. Writing (18) for $n = n_k$, $k = 1, 2, \dots$, and passing with k to infinity we arrive to a contradiction.

Thus, (19) holds for each x and f with $f(x) \neq 0$. Consider now an arbitrary $T \in \mathcal{L}(X)$. Then given any natural number $n \geq 1$ and $x \in X$, $f \in X^*$ with $f(x) \neq 0$, by (7) we have that

$$r_\varepsilon(\varphi(T)\alpha(nx \otimes f)U(nx \otimes f)U^{-1}) = r_\varepsilon(T(nx \otimes f)),$$

and therefore

$$r_{\varepsilon/n}(\alpha(nx \otimes f)(\varphi(T)Ux) \otimes (U^{-1})^*f) = r_{\varepsilon/n}((Tx) \otimes f) \quad (n \geq 1).$$

Letting $n \rightarrow +\infty$, then Lemma 6 (ii) and (19) imply that

$$\frac{|f(Qx)|}{|f(x)|} |((U^{-1})^*f)((\varphi(T)Ux))| = |f(Tx)|,$$

and therefore

$$|f(Qx)| \cdot |f(U^{-1}\varphi(T)Ux)| = |f(Tx)| \cdot |f(x)|. \quad (20)$$

The equality (20) holds for $f(x) \neq 0$ and arbitrary $T \in \mathcal{L}(X)$, and then by continuity it holds for every $x \in X$, $f \in X^*$ and $T \in \mathcal{L}(X)$.

Suppose there exists $x \in X$ such that $\{x, Qx\}$ is linearly independent. There exists then $f \in X^*$ such that $f(x) = 1$ and $f(Qx) = 0$. Using (20), this gives $f(Tx) = 0$ for every $T \in \mathcal{L}(X)$ arriving to a contradiction. Thus $Q = \alpha I$ for some scalar α . By (7) we have $r_\varepsilon(\varphi(\alpha I)\varphi(\alpha I)) = r_\varepsilon(\alpha I \cdot \alpha I)$, and therefore $r_\varepsilon(I) = r_\varepsilon(\alpha^2 I)$. Thus $1 + \varepsilon = |\alpha|^2 + \varepsilon$, which gives $|\alpha| = 1$. Then (20) gives $|f(U^{-1}\varphi(T)Ux)| = |f(Tx)|$ for each f and x with $f(x) \neq 0$, and then by continuity

$$|f(U^{-1}\varphi(T)Ux)| = |f(Tx)| \quad (T \in \mathcal{L}(X), x \in X, f \in X^*).$$

This implies that for each $x \in X$, there exists $\alpha_x \in \mathbb{C}$ of modulus one such that $(U^{-1}\varphi(T)U)(x) = \alpha_x T(x)$. Indeed, we have $U^{-1}\varphi(T)Ux = 0$ if and only if $Tx = 0$, while in the case when they are both nonzero we have that they are linearly dependent: if not, one may choose $f \in X^*$ such that $f(U^{-1}\varphi(T)Ux)$ is 0 while $f(Tx)$ is not. Thus in the second case we have $aU^{-1}\varphi(T)Ux + bTx = 0$ for some nonzero complex constants a and b , and by considering $f \in X^*$ such that $f(Tx) \neq 0$ we get $|a| = |b|$.

Lemma 10 proved at the end of this section implies the existence of a map $\xi: \mathcal{L}(X) \rightarrow \mathbb{C}$ such that $|\xi(T)| = 1$ for each T and

$$\varphi(T) = \xi(T)UTU^{-1} \quad (T \in \mathcal{L}(X)). \quad (21)$$

Let us prove now that we may normalize U in order to have $\|U(x)\| = \|x\|$ for every $x \in X$. To see this, let us observe that by (21), we have $T^2 = 0$ if and only if $\varphi(T)^2 = 0$. Thus given any $T \in \mathcal{L}(X)$ such that $T^2 = 0$, by (14) we have $\lim_{\delta \rightarrow 0} (r_\delta(T)/\sqrt{\delta}) = \sqrt{\|T\|}$ and $\lim_{\delta \rightarrow 0} (r_\delta(\varphi(T))/\sqrt{\delta}) = \sqrt{\|\varphi(T)\|}$. Thus $\lim_{\delta \rightarrow 0} (r_\delta(UTU^{-1})/\sqrt{\delta}) = \sqrt{\|UTU^{-1}\|}$. That $\varphi(\alpha I) = I$ for some unimodular constant α and (7) give $r_\varepsilon(\varphi(R)) = r_\varepsilon(R)$ for each $R \in \mathcal{L}(X)$. Then

$$\begin{aligned} r_{\varepsilon/t}(URU^{-1}) &= r_\varepsilon(\xi(tR)U(tR)U^{-1})/t = r_\varepsilon(\varphi(tR))/t = r_\varepsilon(tR)/t \\ &= r_{\varepsilon/t}(R) \end{aligned}$$

for each $t > 0$, and therefore $r_\delta(URU^{-1}) = r_\delta(R)$ for each $R \in \mathcal{L}(X)$ and each $\delta > 0$. We then conclude that

$$\|UTU^{-1}\| = \|T\| \quad (T \in \mathcal{L}(X), T^2 = 0). \quad (22)$$

For $x \in X$ and $f \in X^*$ such that $f(x) = 0$, putting $T = x \otimes f$ in (22) we obtain that $\|(U^{-1})^*(f)\| \|U(x)\| = \|f\| \|x\|$. Since $\dim X \geq 3$, then for two arbitrary nonzero vectors $x, y \in X$ by considering $f \in X^*$ nonzero such that $f(x) = f(y) = 0$, we obtain that

$$\frac{\|U(x)\|}{\|x\|} = \frac{\|U(y)\|}{\|y\|} \left(= \frac{\|f\|}{\|(U^{-1})^*(f)\|} \right).$$

This implies the existence of $t > 0$ such that $\|U(x)\| = t\|x\|$ for each x in X , and then just replace U by U/t .

Thus Theorem 3 will be proved, once we obtain the following somehow folklore result.

Lemma 10. Let $A, B \in \mathcal{L}(X)$ such that for each $x \in X$, there exists $\alpha_x \in \mathbb{C}$ of modulus one such that

$$A(x) = \alpha_x B(x). \quad (23)$$

There exists then $\alpha \in \mathbb{C}$ of modulus one such that $A = \alpha B$ in $\mathcal{L}(X)$.

Proof. We have $\|A(x)\| = \|B(x)\|$ for each x , and therefore $\ker A$ and $\ker B$ coincide. If $\ker A = \ker B = X$, there is nothing to be proved. If not, denote $M = \ker(A)$ and for the quotient $\hat{X} := X/M$, consider the linear bounded operators $A_1, B_1: \hat{X} \rightarrow X$ given by $A_1(\hat{x}) = A(x)$ and $B_1(\hat{x}) = B(x)$ for each $x \in X$. Observe that, once again, for each $x \in X$ there exists $\alpha_x \in \mathbb{C}$ of modulus one such that $A_1(\hat{x}) = \alpha_x B_1(\hat{x})$. But now A_1 and B_1 are both injective operators.

Fix $\hat{x}_0 \in \hat{X} \setminus \{\hat{0}\}$, and let $\alpha \in \mathbb{C}$ of modulus one such that $A_1(\hat{x}_0) = \alpha B_1(\hat{x}_0)$. Let now $\hat{x} \in \hat{X}$ such that $\hat{x}$ and $\hat{x}_0$ are linearly independent in $\hat{X}$. Let $\alpha_1, \alpha_2 \in \mathbb{C}$ of modulus one such that $A_1(\hat{x}) = \alpha_1 B_1(\hat{x})$ and $A_1(\hat{x} + \hat{x}_0) = \alpha_2 B_1(\hat{x} + \hat{x}_0)$. Then

$$B_1((\alpha - \alpha_2)\hat{x}_0 + (\alpha_1 - \alpha_2)\hat{x}) = 0.$$

The injectivity of B_1 gives $(\alpha - \alpha_2)\hat{x}_0 + (\alpha_1 - \alpha_2)\hat{x} = \hat{0}$, and therefore $\alpha = \alpha_1 = \alpha_2$. Thus $A_1(\hat{x}) = \alpha B_1(\hat{x})$ for each $\hat{x}$ such that $\hat{x}$ and $\hat{x}_0$ are linearly independent, and since $A_1(\hat{x}_0) = \alpha B_1(\hat{x}_0)$ this gives $A_1 = \alpha B_1$. Thus, $A = \alpha B$ in $\mathcal{L}(X)$. $\square$

If instead of (23) we merely suppose that $\{Ax, Bx\}$ is linearly dependent for each $x \in X$, the conclusion of Lemma 10 no longer holds: one can only conclude the existence of $\alpha, \beta \in \mathbb{C}$ and $x_0 \in X, f_0 \in X^*$ such that $\alpha A + \beta B = x_0 \otimes f_0$. (See, for example, [1, Theorem 4.2.9 and Remark 3 at page 88].) If $\{Ax, Bx\}$ is always linearly dependent and B is supposed injective, as in the proof of Lemma 10 we obtain the existence of $\alpha \in \mathbb{C}$ such that $A = \alpha B$. This holds for example when B is the identity operator on X .

3.2 .The finite-dimensional case: proof of Theorem 4

Again, one can easily check that maps of the form (10) and (11) satisfy (9). For the converse, most of the proof for this case has also been carried out in the infinite-dimensional case. Instead of Lemma 10, we shall use the following result.

Lemma 11. (See [8, Theorem 2.1].) Let $n \geq 3$ and suppose $\varphi: \mathcal{M}_n \rightarrow \mathcal{M}_n$ is a map such that

$$\varphi(T_1)\varphi(T_2) = 0 \Leftrightarrow T_1 T_2 = 0 \quad (T_1, T_2 \in \mathcal{M}_n).$$

There exist then a functional $\alpha: \mathcal{M}_n \rightarrow \mathbb{C} \setminus \{0\}$, an invertible matrix $U \in \mathcal{M}_n$ and a field monomorphism η of $\mathbb{C}$ such that

$$\varphi(T) = \alpha(T)UT^\eta U^{-1} \quad (T \in \mathcal{M}_n, \text{ rank}(T) = 1). \quad (24)$$

(By $T^\eta \in \mathcal{M}_n$ we denote the matrix obtained from $T \in \mathcal{M}_n$ by applying η to every entry of it.)

Theorem 5 and lemma 11 imply the existence of α and U such that (24) holds. We shall write in the following the elements of $\mathbb{C}^n$ as column vectors. Let us observe that for $f = [f_1, \dots, f_n]^t \in \mathbb{C}^n$ and $x = [x_1, \dots, x_n]^t \in \mathbb{C}^n$, then $xf^t \in \mathcal{M}_n$ is a rank one matrix, which corresponds to the rank one operator $x \otimes f: \mathbb{C}^n \rightarrow \mathbb{C}^n$ given by $y \rightarrow f(y)x = (f^t y)x = (\sum_{j=1}^n f_j y_j)x$, for $y = [y_1, \dots, y_n]^t \in \mathbb{C}^n$. Let us also observe that for each f and x , we have $(xf^t)^\eta = x^\eta (f^t)^\eta \in \mathcal{M}_n$, and then (24) gives

$$\varphi(xf^t) = \alpha(xf^t)Ux^\eta (f^t)^\eta U^{-1} \quad (f, x \in \mathbb{C}^n)$$

Let us first prove that $T \rightarrow |\alpha(T)|$ is constant on the set of matrices $T \in \mathcal{M}_n$ of rank exactly one. Indeed, if $T_1 = xf_1^t$ and $T_2 = xf_2^t$ for $x, f_1, f_2 \in \mathbb{C}^n$ with x nonzero and $\{f_1, f_2\}$ linearly independent, then for $T = x_0 f_0^t$ with $x_0, f_0 \in \mathbb{C}^n$ nonzero such that $f_1^t x_0 = f_2^t x_0$ and the common value is nonzero, we have $T_1 T = (f_1^t x_0) x f_0^t = (f_2^t x_0) x f_0^t = T_2 T$. Then (9) and (24) give

$$\begin{aligned} r_\varepsilon(\alpha(T_1)\alpha(T)U(T_1 T)^\eta U^{-1}) &= r_\varepsilon(T_1 T) = r_\varepsilon(T_2 T) \\ &= r_\varepsilon(\alpha(T_2)\alpha(T)U(T_2 T)^\eta U^{-1}) \\ &= r_\varepsilon(\alpha(T_2)\alpha(T)U(T_1 T)^\eta U^{-1}) \end{aligned}$$

Since $T_1 T \neq 0$ and η is a monomorphism, then $(T_1 T)^\eta \neq 0$. Thus $U(T_1 T)^\eta U^{-1} \neq 0$, and then part (ii) of lemma 12 below gives $|\alpha(T_1)||\alpha(T)| = |\alpha(T_2)||\alpha(T)|$. That $|\alpha(T)| \neq 0$ gives then $|\alpha(T_1)| = |\alpha(T_2)|$. Analogously, if $T_1 = x_1 f_1^t$ and $T_2 = x_2 f_2^t$ for $f, x_1, x_2 \in \mathbb{C}^n$ with f nonzero and $\{x_1, x_2\}$ linearly independent, then $|\alpha(T_1)| = |\alpha(T_2)|$. Thus if x_1, x_2 in $\mathbb{C}^n$ are linearly independent and f_1, f_2 in $\mathbb{C}^n$ are also linearly independent, we have

$$|\alpha(x_1 f_1^t)| = |\alpha(x_1 f_2^t)| = |\alpha(x_2 f_2^t)|.$$

Now let x_1, x_2 be nonzero in $\mathbb{C}^n$, and f_1, f_2 also nonzero in $\mathbb{C}^n$. Choose $x \in \mathbb{C}^n$ and $f \in \mathbb{C}^n$ such that $\{x, x_1\}, \{x, x_2\}, \{f, f_1\}$ and $\{f, f_2\}$ are respectively linearly independent. By what we just proved, we have

$$\begin{aligned} |\alpha(x_1 f_1^t)| &= |\alpha(x f^t)| \\ &= |\alpha(x_2 f_2^t)|. \end{aligned}$$

Thus, there exists $\delta > 0$ such that $|\alpha(x f^t)| = \delta$ for every $x, f \in \mathbb{C}^n \setminus \{0\}$. For $x = f = e_1 = [1, 0, \dots, 0]^t \in \mathbb{C}^n$, and $z \in \mathbb{C} \setminus \{0\}$ for $T = x f^t$ we have $T^2 = T$ and then by (9),

$$r_\varepsilon(\alpha(zT)\alpha(T)U(zT)^\eta U^{-1}) = r_\varepsilon(zT).$$

Then (6) gives

$$\varepsilon + r(\alpha(zT)\alpha(T)U(zT)^\eta U^{-1}) \leq \varepsilon + \|zT\|,$$

and therefore

$$\delta^2 |\eta(z)| \leq |z| \|x f^t\| \quad (z \in \mathbb{C})$$

Thus the monomorphism η of $\mathbb{C}$ is continuous, and this implies that either $\eta(z) = z$ for every $z \in \mathbb{C}$, or $\eta(z) = \bar{z}$ for every $z \in \mathbb{C}$.

For $f, x \in \mathbb{C}^n$, an integer $k \geq 1$ and $T \in \mathcal{M}_n$ arbitrary, that $r_\varepsilon(\varphi(T)\varphi(kx f^t)) = r_\varepsilon(kT x f^t)$ gives

$$r_{\varepsilon/k}(\alpha(kx f^t)\varphi(T)Ux^\eta \cdot (f^t)^\eta U^{-1}) = r_{\varepsilon/k}(T x f^t).$$

Letting $k \rightarrow \infty$, by Lemma 6 we obtain that

$$\delta |f^t((U^\eta)^{-1}(\varphi(T))^\eta U^\eta x| = |f^t T x| \quad (f, x \in \mathbb{C}^n, T \in \mathcal{M}_n) \quad (25)$$

Where η is either the identity, or the conjugation.

For $T = I_n$ in (25), we have that $f^t x = 0$ implies $f^t((U^\eta)^{-1}(\varphi(I_n))^\eta U^\eta x) = 0$. This implies that x and $(U^\eta)^{-1}(\varphi(I_n))^\eta U^\eta x$ are linearly dependent for every x , and therefore $(U^\eta)^{-1}(\varphi(I_n))^\eta U^\eta = \alpha I_n$ for some complex constant α , which, by (25), is not zero. Thus $\varphi(I_n) = \eta(\alpha)I_n$. That $r_\varepsilon(\varphi(I_n)^2) = r_\varepsilon(I_n)$ gives $|\eta(\alpha)| = 1$, and therefore $|\alpha| = 1$. Putting $T = I_n$ in (25), we obtain that $\delta = 1$. Therefore, if η is the identity then

$$|f^t(U^{-1}\varphi(T)Ux| = |f^t T x| \quad (f, x \in \mathbb{C}^n, T \in \mathcal{M}_n),$$

While if η is the conjugation we obtain that

$$|f^t(\bar{U}^{-1}\overline{\varphi(T)}\bar{U}x| = |f^t T x| \quad (f, x \in \mathbb{C}^n, T \in \mathcal{M}_n).$$

Now exactly as in the final part of the proof of Theorem 3 we obtain in the first case that $\varphi(T) = \xi(T)UTU^{-1}$ for every $T \in \mathcal{M}_n$, where $\xi: \mathcal{M}_n \rightarrow \mathbb{C}$ is a functional satisfying $|\xi(T)| = 1$ for every T and U can be normalized such that $\|Ux\| = \|x\|$ for each $x \in \mathbb{C}^n$, while in the second case we have that $\varphi(T) = \xi(T)U\bar{T}U^{-1}$ for every $T \in \mathcal{M}_n$.

where $\xi: \mathcal{M}_n \rightarrow \mathbb{C}$ is a functional satisfying $|\xi(T)| = 1$ for every T and U can be normalized such that $\|Ux\| = \|\bar{x}\|$ for each $x \in \mathbb{C}^n$.

Thus, all that remains to be proved is the following lemma, which is of interest in its own. (see also [7] .)

Lemma12. *i) If $T_0 \in \mathcal{M}_n$ and $\lambda_0 \in \mathbb{C}$ satisfy $|\lambda_0| < 1$ and $r_\varepsilon(\lambda_0 T_0) = r_\varepsilon(T_0)$, then $T_0 = 0$.*

ii) If $T \in \mathcal{M}_n$ is not zero and $\lambda_1, \lambda_2 \neq 0$ in $\mathbb{C}$ satisfy $r_\varepsilon(\lambda_1 T) = r_\varepsilon(\lambda_2 T)$, then $|\lambda_1| = |\lambda_2|$.

Proof. i) Since the spectral radius is continuous on $\mathcal{M}_n$ (see, for example, [1, Corollary 3.4.5] and $\mathcal{M}_n$ is of finite dimension, by compactness there exists $E_0 \in \mathcal{M}_n$ with $\|E_0\| \leq \varepsilon$ such that $r_\varepsilon(\lambda_0 T_0) = r(\lambda_0 T_0 + E_0)$. Consider then the subharmonic function $\lambda \rightarrow r(\lambda T_0 + E_0)$ on $\mathbb{C}$ (see, for example, [1, Theorem 3.4.7]). For $|\lambda| = 1$, its value at λ is $r(T_0 + \bar{\lambda} E_0) \leq r_\varepsilon(T_0)$, and its value at λ_0 is $r_\varepsilon(\lambda_0 T_0) = r_\varepsilon(T_0)$. Using the maximum principle for subharmonic functions, we obtain that $r(\lambda T_0 + E_0) = r_\varepsilon(T_0)$ for $|\lambda| \leq 1$. In particular, $r(E_0) = r_\varepsilon(T_0)$. That $\|E_0\| \leq \varepsilon$ gives $r(E_0) \leq \varepsilon$ and therefore $r_\varepsilon(T_0) \leq \varepsilon$. By (6) we get $r_\varepsilon(T_0) = \varepsilon$. Then Theorem 1 implies that $T_0 = 0$.

ii) If, for example, $|\lambda_1| < |\lambda_2|$, then for $\lambda_0 = \lambda_1/\lambda_2$ and $T_0 = \lambda_2 T$ we have $|\lambda_0| < 1$ and $r_\varepsilon(\lambda_0 T_0) = r_\varepsilon(T_0)$. Then part (i) gives $T_0 = 0$. Thus $T = 0$, arriving to a contradiction.

Resources

- [1] B. Aupetit, A Primer on Spectral Theory, Springer-Verlag, 1991.
- [2] M. Bendaoud, A. Benyouness, M. Sarih, Preservers of pseudo spectral radius of operator products, Linear Algebra Appl. 489 (2016) 186-198.
- [3] F. Chaitin-Chatelin, A. Harrabi, About definitions of pseudospectra of closed operators in Banach spaces, Tech. Rep. TR/PA/98/08, CERFACS.
- [4] J. Cui, V. Forstall, C.K. Li, V. Yannello, Properties and preservers of the pseudospectrum, Linear Algebra Appl. 436 (2012) 316-325.
- [5] J. Cui, J. Hou, Maps leaving functional values of operator products invariant, Linear Algebra Appl. 428 (2008) 1649-1663.
- [6] J. Cui, C.K. Li, Y.T. Poon, Pseudospectra of special operators and pseudospectrum preservers, J. Math. Anal. Appl. 419 (2014) 1261-1273.
- [7] J. Cui, C.K. Li, Y.T. Poon, Preservers of unitary similarity functions on Lie products of matrices, Linear Algebra Appl. 498 (2016) 160-180.
- [8] J.T. Chan, C.K. Li, N.S. Sze, Mappings on matrices: invariance of functional values of matrix products, J. Aust. Math. Soc. 81 (2006) 165-184.
- [9] J. Globevnik, On vector-valued analytic functions with constant norm, Studia Math. 53 (1975) 29-37.
- [10] J. Globevnik, Norm-constant analytic functions and equivalent norms, Illinois J. Math. 20 (1976) 503-506.
- [11] A. Harrabi, Pseudospectre d'une suite d'opérateurs bornés, RAIRO Modél. Math. Anal. Numér. 32 (1998) 671-680.
- [12] E. Shargorodsky, On the level sets of the resolvent norm of a linear operator, Bull. Lond. Math. Soc. 40 (2008) 493-504.
- [13] E. Shargorodsky, On the definition of pseudospectra, Bull. Lond. Math. Soc. 41 (2009) 524-534.
- [14] J.G. Stampfli, An extreme point theorem for inverses in a Banach algebra with identity, Proc. Camb. Philos. Soc. 63 (1967) 993-994.
- [15] L.N. Trefethen, M. Embree, Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators, Princeton University Press, Princeton, NJ, 2005.
- [16] B. Aupetit and H. du T. Mouton, Spectrum-preserving linear mappings in Banach algebras, Studia Math. 109 (1994), 91-100.
- [17] A. A. Jafarian and A. R. Sourour, Spectrum-preserving linear maps, J. Funct. Anal. 66 (1986), 255-261.
- [18] M. Marcus and B. N. Moyls, Linear transformations on algebras of matrices, Canad. J. Math. 11 (1959), 61-66.