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Semi-Modules Which are E-Piequvelent to Projective Semi-Module

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Abstract

The aim of this study is an expanded study of the concept projective semi module. I projective property was defined for the module in a certain way, and we described it for the semi-module as follows: a semi-module X is projective if for any surjective hom $f: B \to A$ and any hom $g: X \to A$, ∃ a hom h: X→ B s.t f h(X) ⊇ g(X). This concept is closed, as in modules, under direct sum. X is an E-piequvelent semi-module to Y if there are surjective homs h: X→ Y and g: Y→ X. Any semi-module is a projective semi-module if and only if it is an im-projective semi-module and quasi-projective semi-module, where a semi-module M is quasi-projective if, for any surjective hom $f: M \to N$, and any hom g:M→ N , ∃a hom α s.t f α= g.

1. Introduction

In previous research, we studied the concept of rad-projective and strongly rad-projective on semi-modules. This paper will examine a new concept (previously studied on modules[1]), the idea of im-projective, for semi-modules.

Projective semi-modules were studied a lot in the literature ([1], [3], [5], [6], …), but the generalizations of projective semi-modules are less found. As in a previous paper[4], another generalization of projective semi-module will be introduced and studied in this paper. A semi-module X is im-projective if for any surjective hom $f: A \to B$ and any hom $g: X \to A$, ∃a hom h: X→ B s.t f h(X) ⊇ g(X). This concept was studied for the module, and it was proved that the class of im-projective modules is closed under direct sums, and it contains the principal left ideals of a ring R for which $Rx=Rx^2$ and the class of modules which are e-equivalent to projective modules[8]. Some of the results achieved in modules can be proved directly for semi-modules, but other results need to add conditions or weaken them. In the following, left unitary semi-modules over a semi-ring are considered, $S_1 \le S_2$, $S_1 \cong S_2$, S^N, $S_1 \equiv S_2$, denote S_1 sub-semi-module of S_2 , S_1 isomorphic to S_2, the direct sum of N copies of S, and S_1 is e-equivalent semi-module to S_2, respectively.

A semi-module S is strictly an im-projective semi-module if S is an im-projective semi-module but not projective.

The second section will give a background for semi-rings and semi-modules, including the definitions and concepts related to semi-modules that will be needed to fulfill our study,. At the same, The main results will appear in the last section.

2. Definitions and propositions

Definition 2.1. [2] Assume ℵ be a semi-ring. A left ℵ-semi-module A is a commutative monoid $(A, +, 0)$ for which we have a function $\aleph \times A \longrightarrow A$, defined by $(n, a) \mapsto na$ such that $\forall$ n, $t \in \aleph$ and $a, a' \in A$, 1- n$(a + a^{'})$ = na + na'.

2- (n + $t)a$ = na + ta.

3- (n$t)a$ = n(ta).

4- 0_n a = 0_A = n0_A.

If $1_{\aleph}a = a$ holds, then a left ℵ-semi-module A is called unitary.

Definition 2.2.[3] Assume Y be a subset of a left ℵ-semi-module P, then Y is called sub-semi-module of P if Y is closed under addition and scalar multiplication. In this case, it is denoted by Y ≤ P.

Definition 2.3.[3] A sub-semi-module O of an ℵ- semi-module P is called subtractive if $\forall\ a, a' \in$ P $a, a + a' \in$ O implies $a^{'} \in$ O.

Note that 0 and A are always subtractive sub-semi-modules of A.

An ℵ-semi-module A is considered subtractive if all its sub-semi-modules are subtractive.

Definition 2.4.[2] Assume U and W be ℵ -semi-modules.U hom from W to W is a map

φ: U ⟶ W s.t

1- T$(a + a^{'}) = \varphi(a) + \varphi(a^{'})$ and

2- T(na) = n$\varphi(a)$ $\forall\ a, a' \in$ U and $s \in$ ℵ.

For a home of ℵ-semi-modules T: U ⟶ W we define:

1- ker(T) = $\{a \in$ U $|\varphi(a) = 0\}$.

2- T(U) = $\{\varphi(a)|a \in$ U$)\}$.

3- $Im(\varphi)$ =$\{b \in$ W | $b + f(a) = f(a')$ for some a,a'∈ U }.

A hom of ℵ-semi-modules T ∶ U ⟶ W isa

1- Isomorphism if φ is injective and surjective map.

2- Image regular (i-regular), if T(U) = Im(T).

3- Kernel regular (k-regular) if T(a) = T$(a^{'})$ implies

$a + k = a^{'} + k'$ for some, $k, k' \in$ ker(φ).

4- Regular if T is i-regular and k-regular.

<u>**Definition.2.5.[3]:**</u> An element n of ℵ is an additive inverse of m∈ ℵ if and only if n+m=0. And n is unique. The set of all elements of $\mathcal{R}$ having additive inverse is denoted by V(ℵ).

<u>**Definition.2.6.[3]:**</u> A semi-module $\mathcal{A}$ is said to be semi-subtractive if for any

a≠ b in $\mathcal{A}$ there is always some x ∈ $\mathcal{A}$ satisfying b + x = a or some y ∈ $\mathcal{A}$ satisfying a+y = b

<u>**Definition.2.7.[3]p.184:**</u> A sub-semi-module ℵ of a left $\mathcal{R}$ -semi-module $\mathcal{M}$ is a direct summand of $\mathcal{M}$ if

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and only if ∃a sub-semi-module ℵ' of $\mathcal{M}$ satisfying $\mathcal{M}$ = ℵ ⊕ ℵ'.

3. Main results

<u>Definition 3.1</u>. A semi-module X is im-projective if for any surjective hom $f: \mathcal{A} \to \mathcal{B}$ and any hom $g: X \to \mathcal{A}$, ∃a hom h: X→ $\mathcal{B}$ s.t f h(X) ⊇ g(X).

A semi-module $\mathcal{A}$ is said to be strictly an im-projective semi-module if $\mathcal{A}$ is an im-projective semi-module but not projective.

Example 3.2. Assume $\mathcal{R}$ be a semi-ring s.t $\mathcal{R}$ = Y ⊕ X with $\mathcal{R}$ ≅ Y ≅ X (e.g. the endo morphism ring of an infinite dimensional vector space over a field). Since R is not compassumeely reducible ∃, a maximal left ideal K is not a direct summand of $\mathcal{R}$. Then $\mathcal{R}$ ⊕ $\mathcal{R}$ /K e-piequivalent of $\mathcal{R}$. Thus, $\mathcal{R}$ ⊕ $\mathcal{R}$ /K is a cyclic strictly im-projective semi-module, a direct sum of a projective semi-module.

Definition 3.3. A semi-module X is e-piequivalent to a semi-module Y if there are surjective homs from X onto Y and Y onto X. In this case, we write X ≡ Y.

Theorem 3.1 Assume X is a semi-module, then the following are equivalent**:**

(1) X is im-projective semi-module.

(2) ∃ a projective semi-module F, and hom f: F→ X and h: X→F s.t fh(X)=X.

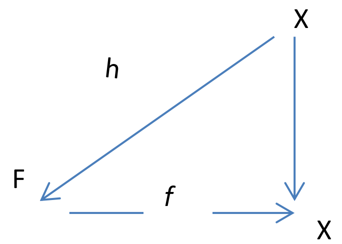
(3) Given any surjective hom f:$\mathcal{B}$ → $\mathcal{A}$ and any hom g: X→ $\mathcal{A}$ ∃a surjective hom k: X→X and a hom h: X→ $\mathcal{B}$ s.t fh(s)=gk(s) ∀ s in X.

<u>Proof:</u>
(1)⟹(2): Assume X is an im-projective semi-module. Since X is a homomorphic image of a free semi-module, say F, assume f: F→X be a surjective hom

consider the diagram, then there exists

h: X→F (F is projective) s.t

fh(X)⊇ 1_X(X)=X (X im-projective). So fh(X)=X.



(2)⟹(3): Assume (2) holds, assume f:$\mathcal{B}$ → $\mathcal{A}$ be surjective, and g: X→ $\mathcal{A}$

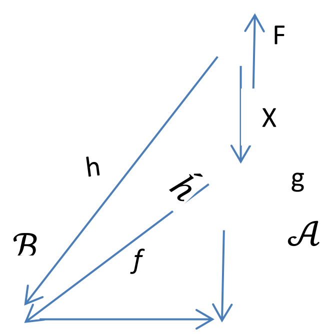
a hom, consider the diagram where F is projective, and α is surjective by (2)

∃β:X→F s.t αβ(X)=X, assume k= αβ:X→ X

Since F is projective, ∃h: F→ $\mathcal{B}$ s.t

fh=gα, put hβ= h́: X→ $\mathcal{B}$

f(h́(s))=f(h(β(s)))=g(α(β(s)))=g(k(s)) ∀ s∈X.



(3) ⟹(1): Consider the diagram

f:$\mathcal{B}$ → $\mathcal{A}$ surjective hom ; g:X→ $\mathcal{A}$ any hom

$$\begin{tikzcd} & X \arrow[d, "k"] \\ & X \arrow[dl, "h"'] \arrow[d, "g"] \\ \mathcal{B} \arrow[r, "f"] & \mathcal{A} \end{tikzcd}$$

then by (3) ∃a surjective k:X →X and

a hom h:X → $\mathcal{B}$ s.t fh(s)=gk(s)

∀ s in X, we must prove that X is im-projective.

Since k is surjective, for every s in X ∃x in X s.t k(x)=s and fh(X)=gk(X)=g(X) and so fh(X)=g(X).

Hence, X is an im-projective semi-module.

Proposition 3.2. Assume X is an im-projective calculative semi-module. If Y is semi-subtractive semi-module and f:Y →X a surjective hom, with ker(f) ⊆V(Y), then Y=$\overline{X}$+ ker(f) where $\overline{X}$ is a homomorphic image of X (Hence $\overline{X}$ is e-piequivalent to X).

Proof: By taking $\mathcal{B}$ =Y, $\mathcal{A}$=X, g=1_X in Theorem 3.1(3), ∃k:X →X and

h:X →Y homs s.t f(h(s)))=k(s) ∀ x∈X.

Then f(h(X)))=k(X)=X and f^{-1}(f(h(X)))= f^{-1}(X)=Y, assume $\overline{X}$=h(X).

It is clear that $\overline{X}$+ ker(f) ⊆Y.

Assume y∈ Ythen f(y) ∈f($\overline{X}$) ⟹ f(y)= f(x) for some x∈ $\overline{X}$.

y=x+ j or x=y+ j(Y is semisubtractive), in any case j∈ ker(f) (X is cancellative). that is, y=x+ j or y=x-j (ker(f) ⊆V(Y)). Hence y∈ $\overline{X}$+ ker(f).

Therefore Y=$\overline{X}$+ ker(f), where $\overline{X}$=h(X), and f(Y)= X,

then $\overline{X}$ is e-piequivalent to X. ■

Proposition3.3: Assume X is a semi-module; consider the following statements:

(1) For every surjective homs f: Y→X; Y=$\overline{X}$+ker(f) where $\overline{X}$ is a homomorphic image of X (Hence $\overline{X}$ is e-piequivalent to X).

(2) ∃a projective semi-module F s.t F=$\mathcal{M}$+K where $\mathcal{M}$ is a homomorphic image of F/K and X is e-piequivalent to $\mathcal{M}$ (Hence $\mathcal{M}$ is e-piequivalent to F/K.

(3) ∃a projective semi-module F and g∈ End(F) s.t g^2(F)=g(F) and X is e-piequivalent to g(F).

(4) ∃a projective semi-module F and f, g ∈ End(F) s.t f^2h=f and X is e-piequivalent to f(F).

Then: (1) ⟹(2), (2) ⟹(3)and (3) ⟹(4)

Proof:

(1) ⟹(2): By (1), if we consider X as a homomorphic image of a free semi-module

say F, (which is projective) and f: F→X is a surjective hom then,

F=$\overline{X}$+ker(f), put $\overline{X}$= $\mathcal{M}$ and ker(f)=K, X=f(F)=f($\mathcal{M}$)+f(K)= f($\mathcal{M}$)+0 = f($\mathcal{M}$) then, X e-piequivalent to $\mathcal{M}$. On the other hand F/ker(f)≅X, so $\mathcal{M}$ e-piequivalent to F/K .

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(2) ⟹(3): Assume (2) holds, assume F= $\mathcal{M}$ +K, where F is projective, $\mathcal{M}$ is e-piequivalent of X and X e-piequivalent of F/K. Take γ: F→ F/K the natural map,

α: F/K→ $\mathcal{M}$ a surjective hom (since $\mathcal{M}$ e-piequivalent F/K).

Assume g= αγ: F→ $\mathcal{M}$ ≤ F, then g∈End(F).

Now, g(F)=g($\mathcal{M}$)+g(K), where g(K)= α(γ(K))= α($\bar{0}$)=0.

So, g(F)=g($\mathcal{M}$)= $\mathcal{M}$, hence g^2(F)=g($\mathcal{M}$)=g(F).

(3) ⟹(4): Put f=g in (3) and h=1_F. ■

Theorem 3.4: Assume X is a semi-module, then the following are equivalent:

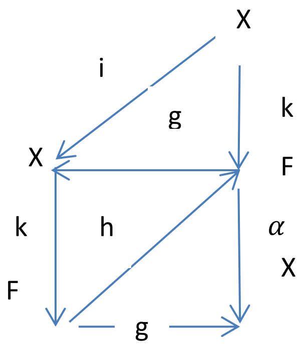
(1) ∃ a projective semi-module F and hom, f:F→ X , h:X→F s.t fh(X)=X
(2) ∃a projective semi-module F and f, g ∈ End(F) s.t f^2h=f and X is e-piequivalent to f(F).

(1) ⟹(2): Assume F is projective semi-module and g:F→ X, k:X→ F

Be hom s.t g(k(X))=X…(∗)

Note that gk is onto, so g is onto take $\alpha = gkg\colon F \to X$

which is surjective.



Consider the diagram, since F is projective

∃h:F→ F s.t $\alpha h = g \ldots (\ast\ast)$

Take f=kg∈End(F), then:

$f^2h = kgkgh = k\alpha h = kg = f$ by(∗∗∗)

Also, f(F)=k(g(F))=k(X), then

k:X→ $k(X) = f(F)$ is surjective and g(f(F))=g(k(X))=X by(∗)

So g:f(F) → X is surjective, then

X is e-piequivalent to f(F).

(2) ⟹(1): Assume k:X→f(F) and g: f(F) →X be surjective hom.

Assume j=gf: F →X, then j(k(X))=gf(f(F))=gf^2(F)=gf(F)=j(F)=X. ■

Lemma 3.5. If $\mathcal{A}$ is im-projective semi-module and $\mathcal{A}$ E-piequvelent of X, then X is im-projective.

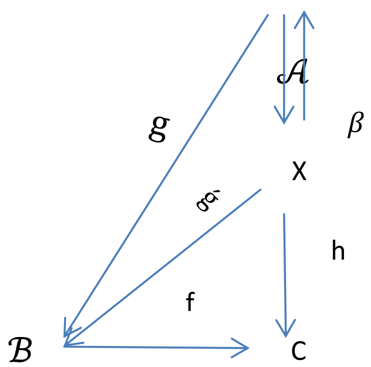
Proof: Consider the diagram:

By assumption, there exists

α: $\mathcal{A}$ ⟶X and β:X ⟶ $\mathcal{A}$ are surjective

Where $\mathcal{B}$, C are any two semi-modules,

f a surjective map and h is a hom.



Since $\mathcal{A}$ is im-projective, ∃

g: $\mathcal{A} \longrightarrow \mathcal{B}$ s.t $f(g(\mathcal{A})) \supseteq h\big(\alpha(\mathcal{A})\big) = h(X)$.

Assume $\acute{g}=g\beta$. Then $f(\acute{g}(X))=f(g(\beta(X)))=f(g(\mathcal{A})) \supseteq h(X)$. Therefore

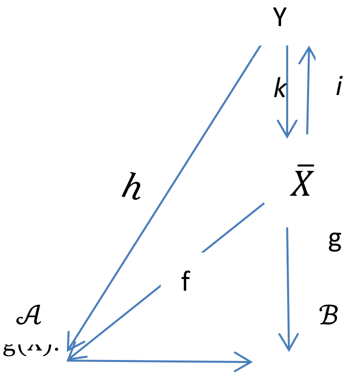
X is im-projective. ■

Theorem 3.6:

Assume X is a semi-module. If for every projective semi-module Y and

surjective hom, f: Y→X, we have $Y=\bar{X}+\ker(f)$ where

$\bar{X} \equiv X$, then X im-projective semi-module.



<u>Proof:</u> Consider the diagram. By assumption

$Y = \bar{X}+\ker(f)$ where $\bar{X} \equiv X$, since Y projective

semi-module then, then ∃h: Y→ $\mathcal{A}$

hom s.t fh= gk but α=hi, then

$f(\alpha(\bar{X}))=f(h(\bar{X}))=gk(\bar{X})= g(k(\bar{X})+k(\ker(k))=g(\bar{X}+\ker(f)) = g(Y) \supseteq g(\mathcal{A})$.

Therefore $\bar{X}$ is im-projective, since $\bar{X} \equiv X$ by (Lemma 3.5.)

X is im-projective. ■

<u>Proposition.3.7.</u> If X is an im-projective cancellative semi-module, then there exist elements $\{x_i : i \in I\}$ in X and homs $\{q_i : i \in I\}$ in Hom(X, $\mathcal{R}$) s.t for each y ∈ X ∃s∈ X, depending on y, s.t $y = \sum x_i q_i(s)$ and $q_i(s)= 0$ ∀ but finitely many i ∈I.

Proof: Since X is im-projective, then by (Theorem.3.1.(2)), take F free(projective) semi-module with f: F→X surjective and h: X→ F hom, then fh(X)=X……..(1). Assume y in X from(Theorem.3.1.(3)) ∃s in X s.t fh(s)=y. Then h(s) in F , then $h(s)=\sum r_i t_i$ where $\{t_i : i \in I\}$ is a basis for F and r_i In the semi-ring $\mathcal{R}$.

Then $y=fh(s)=f(\sum r_i t_i)= \sum r_i f(t_i)=\sum q_i(s)\, x_i$, (take $f(t_i)=x_i$) where $\{x_i : i \in I\}$ } is a set of generators for X, $r_i=q_i(s)$ where q_i:X→ $\mathcal{R}$ and defined by $q_i(x)=c_i$ ∀ x in X where $h(x)= \sum c_i t_i$ and c_i in $\mathcal{R}$. ■

Lemma 3.8. assume $\mathcal{R}$ be a semi-ring , x ∈ $\mathcal{R}$, then

1) If $\mathcal{R} = \mathcal{R}x + \mathrm{ann}_R(x)$, then $\mathcal{R}x = \mathcal{R}x^2$
2) If $\mathcal{R}$ is cancellative, semi-subtractive, and $\mathrm{ann}_R(x) \subseteq V(\mathcal{R})$ with $\mathcal{R}x=Rx^2$ then $\mathcal{R}= \mathcal{R}x + \mathrm{ann}_R(x)$.

Proof(1): Clear

Proof(2):

$\mathcal{R}x = \mathcal{R}x^2$ implies $x=tx^2$ for some t∈ $\mathcal{R}$, since $\mathcal{R}$ is semisubtractive, then ∃h∈ $\mathcal{R}$ s.t 1=tx+h or 1+h=tx. Multiplying by x from right, and using cancellation property by cased ring $x=tx^2$, we get t∈ $\mathrm{ann}_R(x) \subseteq V(\mathcal{R})$.

Hence, we can write $1=tx \mp h \in \mathcal{R}x + \mathrm{ann}_R(x)$.

Therefor $\mathcal{R}= \mathcal{R}x + \mathrm{ann}_{\mathcal{R}}(x)$. ■

Proposition 3.9. Assume X is a cyclic $\mathcal{R}$-semi-module. Then X is im-projective if and only if X is e-piequivalent to a principal left ideal $\mathcal{R}x$ for some $x \in \mathcal{R}$ such that

$\mathcal{R}x = \mathcal{R}x^2$.

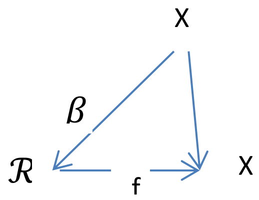
Proof: Assume that X=$\mathcal{R}$m be im-projective, f:$\mathcal{R} \to X$ define as f(r)= rm. Then f is surjective, hence fβ(X)⊇X and fβ(X)=X, then

β(X) = $\mathcal{R}$x [x= β(m)] and fβ(X)=X, then f(β(Rm))= $\mathcal{R}$ (f(β(m))

=$\mathcal{R}$xm, So $\mathcal{R}$m=$\mathcal{R}$xm on the other hand β(X) is a homomorphic

image of X. Therefore, X is e-piequivalent to β(X)

which is an ideal of $\mathcal{R}$. β(X)= β($\mathcal{R}$m)= $\mathcal{R}$β(m) if β(m)=x, then β(X)= $\mathcal{R}$x.



kerf={$r \in \mathcal{R}: f(r) = rm = 0$}, if β(rm)=0, then r $\beta(m) = 0$ and so rx=0 thus r∈ $ann_R(x)$, thus kerf⊆ $ann_R(x)$ so by (proposition 3.8) $\mathcal{R}=\mathcal{R}x+ann_{\mathcal{R}}(x)$ thus $\mathcal{R}x = \mathcal{R}x^2$.

Conversely, Assume that X is e-piequivalent to $\mathcal{R}$x where $\mathcal{R}x = \mathcal{R}x^2$.

Assume g:$\mathcal{R} \to \mathcal{R}$ define as g(r)= rx, then g∈ $End_{\mathcal{R}}(\mathcal{R}_{\mathcal{R}})$and $g^2(\mathcal{R})=\mathcal{R}x^2$ that is g($\mathcal{R}$)= $g^2(\mathcal{R})$, where $\mathcal{R}_{\mathcal{R}}$ projective semi-module, and X is e-piequivalent to

g($\mathcal{R}$)= $\mathcal{R}$x , then by(Lemma 3.5), (Theorem 3.4) and (Theorem 3.1)

X is im-projective. ■

__**Corollary .3.10.**__ Assume $\mathcal{R}$ be a semi-ring with x∈ $\mathcal{R}$ s.t $\mathcal{R}$x is a maximal left ideal. If $ann_R(x) \not\subseteq \mathcal{R}x$, then $\mathcal{R}$x is im-projective.

__**Proof:**__ by assumption, we have $\mathcal{R} = \mathcal{R}x + ann_R(x)$. By 3.10(1) $\mathcal{R}x = \mathcal{R}x^2$. By 3.9 $\mathcal{R}$x is im-projective.
■

__**Corollary .3.11.**__

If R is a p.p. semi-ring (every principal left ideal is projective), then every cyclic im-projective semi-module is e-piequivalent to a direct summand of $\mathcal{R}$.

__**Proof:**__ Assume X be cyclic $\mathcal{R}$.semi-module and im-projective, then by 3.9 X is e-piequivalent to $\mathcal{R}$x for some x∈ $\mathcal{R}$ with $\mathcal{R}x = \mathcal{R}x^2$.

By assumption $\mathcal{R}$x is a projective; hence $\mathcal{R}$x is a direct summand of $\mathcal{R}$ (since

$\mathcal{R} \to \mathcal{R}x$;r⟼rx is surjective implies kerf $\leq^{\oplus}$R[3].

$\mathcal{R}$/kerf≅ $\mathcal{R}$x and $\mathcal{R}$=I⊕kerf for some I≤ $\mathcal{R}$ implies $\mathcal{R}$/kerf≅ I implies $\mathcal{R}$x≅ I implies X≡I$\leq^{\oplus}$ $\mathcal{R}$.
■

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