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Consensus Evaluation: A Practical 4D Geometry Method for 7-Point Likert Data

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Abstract

Consensus measures are commonly used in research and typically rely on Likert scales, focusing on the 5-point Likert scale. One of the main attributes of this scale is that it is designed to work for any size. However, many of the researchers used the 5-point Likert scale. While some researchers have attempted to generalize consensus measures to work in two-dimensional space, this approach remains complex and challenging. In this work, we generalize the measure of consensus to work for seven Likert scales using the computational geometry of four-dimensional concepts. A numerical example and some related concepts are provided.

Keywords: Optimization, consensus measure, Likert scale, 4 – 4-dimensions, probability.

1. Introduction

The fact that in many cases, it is not advantageous for the society at large if the decision is imposed by a subset of people—even if this subset includes more than half of the collective—as there may be other options that the other members more accept and that increases the level of satisfaction overall—means that consensus-based decision-making is crucial for achieving good group performance and, as a result, great individual satisfaction [4].

There are two main challenges in measuring the consensus within the seven Likert scales. Firstly, as in many other fields of study, the measure of consensus depends on opinions, feelings, or beliefs. That means we do not have the correct answer that can be "exact" to get or to compare with. Secondly, the difficulties of working in more than two dimensions are directly proportional to higher dimensions [11]. In other words, there are more challenges when the work depends on opinions and is in the 4-D space [5, 7]. Various researchers many years ago worked to overlap the way of turning opinion and human thought into numbers [6]. Scales are the most common methods or approaches for changing the ideas feeling to numbers. Likert, Guttman, and Borg scales were developed for this purpose [2, 10]. Initially, the Likert scale data was used in two different ways: interval and ordinal [17].

This work depends on collecting the data on the ordinal Likert scale. Even though the five-Likert scale is the most used, the seven-Likert scale is also used frequently in many studies. Seven-point Likert scale is more efficient than the five-point scale [14]. More than one researcher believes that the more freedom you give the users in the survey, the more accurate results you can get [1].

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Although finding an accurate measure in two or three-dimensional space has some difficulties, there are several works on measuring consensus in two or three-dimensional space [16]. One of the most common methods is based on clustering algorithms, where the data points representing the group members are clustered together based on their similarity in a two or three-dimensional space [8]. For instance, in social network analysis, consensus can be measured by analyzing the clustering of nodes within a network [3]. Network nodes can be visualized in two-dimensional or three-dimensional space to show their connection. The level of clustering within the network can be assessed using techniques like modularity, which gauges how nodes are organized into communities [9].

Consensus-based decision-making is better for achieving group satisfaction than decisions imposed by a majority. However, measuring consensus can be challenging, especially when dealing with subjective data like opinions on a 7-point Likert scale. While there are established methods for measuring consensus in 2 or 3 dimensions, these methods become more complex in higher dimensions. The paper then explores existing techniques for measuring consensus in different fields, including social network analysis, geographical information systems, and game theory. These techniques examine how data points representing individuals or groups are clustered or distributed in their respective spaces[18].

Another way to gauge agreement in two or three spaces involves examining data. For instance, consensus can be assessed in geographical information systems by studying how data points from regions are distributed [12]. One method is to calculate autocorrelation, which determines the similarity between data points. Moreover, in game theory, consensus can be evaluated by studying the equilibrium positions of games played in two or three spaces [13]. These equilibrium points signify the stage where all players reach an understanding regarding the game's outcome.

2. Mathematical Base

For $n = 7$, it is recalled that the set $D(\mu, \sigma^2)$ can be defined by:

$$\sum_{i=1}^{7} i^j \cdot p_i = j^{2-j} \cdot \mu^j + j^{2-j}(k-1)\sigma^2 \qquad , k = 0,1,2$$

Without losing generalizing, we replace $p_4 = x$, $p_5 = y$, $p_6 = z$, and $p_7 = w$, then we can determine the three equations above for p_i Where $i = 1,2,3$ in terms of x, y, z, and w. .

$$\begin{bmatrix} 2p_1 \\ p_2 \\ 2p_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -5 \\ -1 & -1 & 4 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} \sigma^2 \\ \mu^2 \\ \mu \end{bmatrix} + \begin{bmatrix} -2 & -6 & -12 & -20 \\ 3 & 8 & 15 & 24 \\ -6 & -12 & -20 & -30 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} \tag{2}$$

Since all the probabilities p_1, p_2, and p_3, are positive and limitations of the subspace $D(\mu, \sigma^2)$ can be specified by:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \begin{bmatrix} 2 & 6 & 12 & 20 \\ 3 & 8 & 15 & 24 \\ 6 & 12 & 20 & 30 \end{bmatrix} \begin{bmatrix} \leq \\ \geq \\ \leq \end{bmatrix} \begin{bmatrix} 1 & 1 & -5 \\ -1 & -1 & 4 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} \sigma^2 \\ \mu^2 \\ \mu \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} \tag{3}$$

To simplify (3), let us define t, u, and r as below:

$$\begin{aligned} t &= h_m(\sigma^2) = \frac{\sigma^2 + \mu^2 - 3\mu + 2}{2} \\ u &= u(\mu) = \frac{(\mu - 1)}{2} \\ r &= r(\mu) = \max\{2\mu - 5, \mu - 2, 0\} \end{aligned} \tag{4}$$

In the notation of (4), we can rewrite (3) as:

$x, y, z, w \geq 0.$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \begin{bmatrix} 2 & 6 & 12 & 20 \\ 3 & 8 & 15 & 24 \\ 6 & 12 & 20 & 30 \end{bmatrix} \begin{bmatrix} \leq \\ \geq \\ \leq \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} t \\ u \\ r \end{bmatrix} \tag{5}$$

Consequently, $D(\mu, \sigma^2)$ can be reduced to the set of all pairs (x, y, z, w) in the $4-$Dimensions space that satisfy (5). Depending on the symmetry of the means concerning the midpoint $\mu = 4$, in the subsequent discussions, we will restrict the range of the mean to $1 \leq \mu \leq 4$. The leftover interval $(4 \leq \mu \leq 7)$ can be preserved as the symmetric reflection of $1 \leq \mu \leq 4$.

Since $1 \leq \mu \leq 4$ and by using the statement of the variance boundaries for each given mean, see [1], we can calculate the boundaries of t (the minimum and the maximum) by substituting each end of variance of $t = h(\sigma^2)$. In our case, for $n = 7$, to find max t ($\max_t$) we have:

$$\begin{aligned} \max_t &= h(U_\mu) \\ &= h\big((\mu - 1)(7 - \mu)\big) \\ &= \frac{-\mu^2 + 8\mu - 7 + \mu^2 - 3\mu + 2}{2} \\ &= \frac{5\mu - 5}{2} \\ &= 5\left(\frac{\mu - 1}{2}\right) \\ &= 5u \end{aligned}$$

Then, the minimum t can be obtained by substitute $L_1(\mu), L_2(\mu)$, and $L_3(\mu)$ only due to the similarity we mentioned above. So we have $\min_t = \max\{2m - 5, m - 2, 0\}$

That will imply $\max_t = h_\mu(U_\mu) = 5u$, and $\min_t = h_\mu(L_\mu) = r$. Therefore, $t = h_\mu(\sigma^2)$ is a linear (one-to-one) mapping of the interval $[L_\mu, U_\mu]$ onto $[r, 5u]$. The inverse of $t = h_\mu(\sigma^2)$ is also a linear mapping of $[r, 5u]$ onto $[L_\mu, U_\mu]$ given by $\sigma^2 = h_\mu^{-1}(t) = 2t - \mu^2 + 3\mu - 2.$

To find the area $D(\mu, \sigma^2)$, solve (6) for w as follows:

$$g_1(x, y, z) = \frac{1}{10}\left(\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\begin{bmatrix} t \\ \mu \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right), g_2(x, y, z) = \frac{1}{24}\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\begin{bmatrix} t \\ \mu \end{bmatrix} - \begin{bmatrix} 3 \\ 8 \\ 15 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right), \text{ and } g_3(x, y, z) = \frac{1}{15}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} t \\ \mu \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$$

Since $1 \leq \mu \leq 4$ by assumption, we have $x \geq 0$, and $t \geq r \geq 2m - 5$. These conditions imply.

$$\begin{aligned} g_2(x, y, z) - g_1(x, y, z) &= \frac{1}{120}\left(\begin{bmatrix} -2 \\ 7 \\ -19 \end{bmatrix}\begin{bmatrix} t \\ \mu \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) \\ &\leq \frac{m - 4}{20} \leq 0 \end{aligned}$$

and that means we get $g_2(x, y, z) \leq g_1(x, y, z)$. Hence, we get the following set relationship.

$$D(\mu, \sigma^2) = D_1(\mu, \sigma^2) - D_2(\mu, \sigma^2)$$

where

$$D_1(\mu, \sigma^2) = \{(x, y, z, w) | \; g_2(x, y, z) \leq w \leq g_3(x, y, z), \quad 0 \leq x, y, z \leq 1\}$$

and

$$D_2(\mu,\sigma^2) = \{(x,y,z,w)|\ g_1(x,y,z) \le w \le g_3(x,y,z), \qquad 0 \le x,y,z \le 1\}$$

In other words, the volume between two hyperplanes $D_1(\mu,\sigma^2)$ and $D_2(\mu,\sigma^2)$ In 4 - dimensions space has been determined.

3. The intersection in Four Dimension

The work in four-dimensional space is more complicated and not straightforward to visualize than in two and three dimensions. However, to understand the way of working in four dimensions, one must first know exactly how it works in one, two, and three dimensions. The figures below are the shortest way to see how we can visualize the graph in all these dimensions.

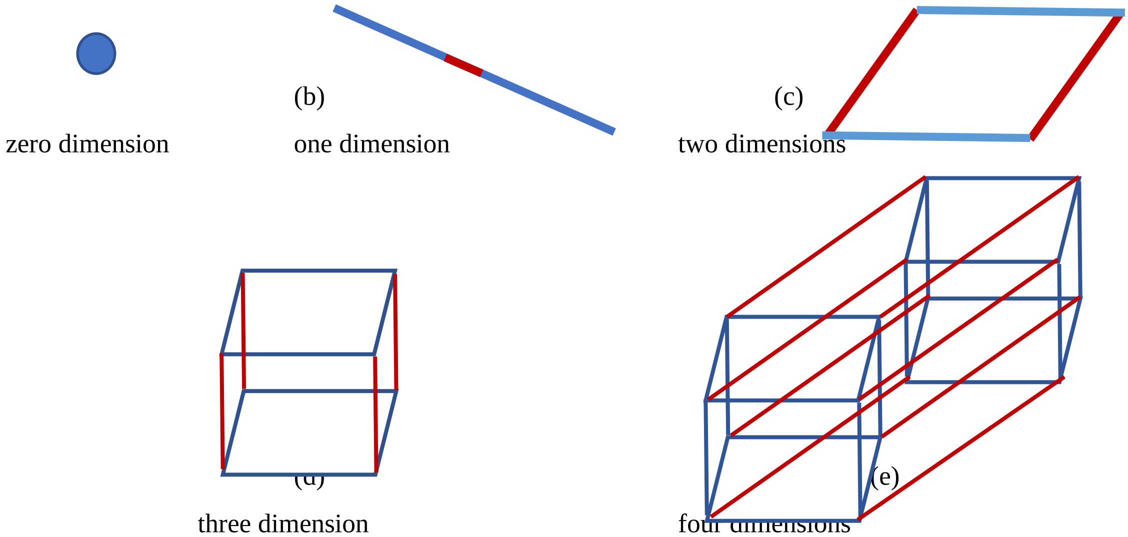

Figure (1)

The graph in zero to four dimensions

Theoretically, the four-dimensional space $\mathbb{R}^4$ is the set of ordered quadruples of real numbers:

$$\mathbb{R}^4 = \{(x,y,z,w): x,y,z,w \in \mathbb{R}\}$$

One of the new objects we need to know in four dimensions is a Hyperplane. A hyperplane in $\mathbb{R}^4$ is the set of vectors $s = (x,y,z,w)$ that satisfy an equation of the form:

$$ax + by + cz + dw = 0$$

Where $a, b, c,$ and d are not all zero fixed real numbers. Note that the general hyperplane is composed of vectors. v_1, v_2, v_3, not lying in a common plane, orthogonal to the average vector $n = (a,b,c,d)$, which must be nonzero. The intersection of two hyperplanes:

$$a_1x + b_1y + c_1z + d_1w = a_2x + b_2y + c_2z + d_2w = 0$$

with non-symmetric normal vectors $n_1 = (a_1,b_1,c_1,d_1)$and $n_2 = (a_2,b_2,c_2,d_2)$ is a plane. For instance,

$$(x = 0) \cap (z = 0) = \{(0,y,0,w):\ y,w \in\ R\}$$

It is the $yw-$plane.

The intersection of any two planes in three dimensions is a line or plane, but the problem is different in four dimensions. The intersection of two planes in four dimensions usually, is a point 0. Imagine that, say, for example, a point on the $yz-$ plane has $x = w = 0$, while a point in the $xw-$ plane has $y = z = 0$, and that's mean the only mutual point is $(0, 0, 0, 0)$.

If we want to understand how all the points represent and work in four dimensions, let go now with more details using matrix concepts. Any point (x, y, z, w) in the intersection satisfies all equations in (4). In matrix languages, any point in the intersection should satisfy the matrix equation.

$$Av = 0$$

In other words, the intersection of four hyperplanes is the kernel of the matrix A, where.

$$\ker A = \{v \in \mathbb{R}^4 : Av = 0\}$$

By using the definition of the kernel, we can now say the intersection of two planes, or four hyperplanes, is exactly one point when the kernel is trivial, i.e., $\ker A = 0$. To be clearer, there are five possible dimensions for any intersection, for general 4×4 matrix A, of $\ker A = 0$:

$$\ker A = 0 \qquad \dim \ker A = 0$$

$$\ker A = \text{a line} \qquad \dim \ker A = 1$$

$$\ker A = \text{a plane} \qquad \dim \ker A = 2$$

$$\ker A = \text{a hyperplane} \qquad \dim \ker A = 3$$

$$\ker A = \mathbb{R}^4 \qquad \dim \ker A = 4$$

4. Computing Index of Agreement

Once we determine the area of $D(\mu; \sigma^2)$, the rest of the work is the numerical integration to get the measure of consensus. In [1], the consensus measure is stated as follows:

$$\Phi(\sigma^2|\mu) = \frac{\int_r^{\tau} g(t)dt}{\int_r^{\tau_{max}} g(t)dt} \qquad (*)$$

Where:

$\tau = \frac{1}{2} 2 (\sigma^2 + \mu^2 - 3m + 2)$, and $\tau_{max} = 5u$ is the max t.

Mushtaq and Darrah presented a new algorithm to find the index of disagreement [2]—the index's integrations determined using Simpson's method. Any numerical integration method can be used to get the integration value. This algorithm still works for any dimensions since, in the end, we only have the "hyper-volume" values representing a function in three dimensions. $w = f(x, y, z)$. Indeed, you can store all the volume (or hyper-volume) values and then determine the curve fitting to this data using any curve fitting methods or software packages. Once you have the function that best fits with the data, take the integration from zero to one of your functions. Now, as we are done with the problem of how to play with our hyper–volume values, the rest of the work is to get the consensus by writing the algorithm steps of finding the index of disagreement.

For a given mean μ and variance σ^2, the following algorithm is used to determine the consensus values.

Algorithm

The inputs of this algorithm are mean μ and variance σ^2. While the output is: the index of disagreement and the consensus value.

i. If $\mu > 4$, then $\mu = 8 - \mu$.
ii. Set N. {N is any large number}.

iii. Determined $t;\ u\ and\ r$.

iv. Determined equation (*) using and numerical integration method, say Phi.

v. $Consensus = 1 - Phi$.

Note that the accurate result of the integration method plays an essential role in the result of the consensus measure. Moreover, it would help if you noticed that the technique should work with multi-dimension space.

5. Numerical Example

To ensure that all the theoretical steps work fine, we examined them using a real numerical example. The data used from random mean and variance values that could cover all cases of the mean and variance. All the results of the measure of consensus are for $n = 7$.

Table (1) shows the measure of consensus with selected values of mean μ close to the left end. At the same time, the variances are close to the minimum and maximum variance concerning this mean. Depending on the statement in [1], the minimum variance when $\mu = 1.1$ is 0.19, while the maximum is 0.39.

Mean μ	Variance σ^2	Consensus
1.1	0.09	1
1.1	0.34	0.5
1.1	0.59	0

$Table\ 1$: $Different\ variances\ with\ left\ end\ mean$

Notice that these results are precisely the same when you have the mean $\mu = 6.9$ due to the similarity of the mean and the variances.

Table (2) offers the measure of consensus when the mean $\mu = 4$, which is the mean in the middle of the mean range. The variances are the same as in case one, close to the minimum and maximum variance concerning this mean ($\sigma^2 = 0, and\ \sigma^2 = 9$).

Mean μ	Variance σ^2	Consensus
4	0.00	1
4	4.50	0.52
4	9.00	0

$Table\ 2$: $Different\ variances\ with\ mid\ of\ mean$

In the third table, we choose different random values to make sure that it will work in any number for mean form $\mu = 1\ to\ \mu = 7$ and any depends on variance values.

Mean μ	Variance σ^2	Consensus
2.25	2.75	0.308
4	2.25	0.875
5.5	4.58	0.17

$Table\ 3$: $Different\ mean\ with\ variances\ values$

Notice that all the results above are reasonable and acceptable, specifically if we compare these results with similar cases for $n = 5$ in [2]. However, the comparison should be with consideration that for bigger n, the maximum value of the variance will not be the same. That means a different range of differences in each case.

6. Conclusion

The proposed four-dimensional approach offers a more flexible and adaptable method for measuring consensus across a range of Likert scales, which could have important implications for future research. Although working in more than three dimensions is problematic because it is more complicated, this paper applied the new consensus measure in four dimensions. Even though the base of the equations looks like the theoretical foundations in [19], it's so hard to try to generate the work to make it work for the seven Likert scales. Consequently, this work used a different approach to determine all the data in the above examples. To make sure that the results are accurate, two other programs were used. The first is in MATLAB, and the second is Visual Basic (VB).

7. Declaration

Not Applicable

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